Principles of Program Analysis:

Abstract Interpretation

Transparencies based on Chapter 4 of the book: Flemming Nielson, Hanne Riis Nielson and Chris Hankin: Principles of Program Analysis. Springer Verlag 2005. ©Flemming Nielson & Hanne Riis Nielson & Chris Hankin.

A Mundane Approach to Semantic Correctness

Semantics:

$$p \vdash v_1 \leadsto v_2$$

where $v_1, v_2 \in V$.

Note: $\leadsto$ might be deterministic.

Program analysis:

$$p \vdash l_1 \rhd l_2$$

where $l_1, l_2 \in L$.

Note: $\rhd$ should be deterministic:

$$f_p(l_1) = l_2.$$

What is the relationship between the semantics and the analysis?

Restrict attention to analyses where properties directly describe sets of values i.e. *"first-order'" analyses* (rather than *"second-order" analyses*).

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Example: Data Flow Analysis

Structural Operational Semantics:

Values: $V = \mathbf{State}$

Transitions:

$$S_\star \vdash \sigma_1 \rightsquigarrow \sigma_2$$

iff

$$\langle S_\star, \sigma_1 \rangle \rightarrow^* \sigma_2$$

Constant Propagation Analysis:

Properties: $L = \widehat{\mathbf{State}}_{\mathsf{CP}} = (\mathbf{Var}_\star \rightarrow \mathbf{Z}^\top)_\bot$

Transitions:

$$S_\star \vdash \widehat{\sigma}_1 \rhd \widehat{\sigma}_2$$

iff

$$\widehat{\sigma}_1 = \iota$$
$$\widehat{\sigma}_2 = \bigsqcup\{\mathsf{CP}_\bullet(\ell) \mid \ell \in \mathit{final}(S_\star)\}$$
$$(\mathsf{CP}_\circ, \mathsf{CP}_\bullet) \models \mathsf{CP}^=(S_\star)$$

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Example: Control Flow Analysis

Structural Operational Semantics:

Values: $V = \mathbf{Val}$

Transitions:

$$e_\star \vdash v_1 \rightsquigarrow v_2$$

iff

$$[\,] \vdash (e_\star\ v_1^{\ell_1})^{\ell_2} \longrightarrow^* v_2^{\ell_2}$$

Pure 0-CFA Analysis:

Properties: $L = \widehat{\mathbf{Env}} \times \widehat{\mathbf{Val}}$

Transitions:

$$e_\star \vdash (\widehat{\rho}_1, \widehat{v}_1) \rhd (\widehat{\rho}_2, \widehat{v}_2)$$

iff

$$\widehat{\mathsf{C}}(\ell_1) = \widehat{v}_1$$
$$\widehat{\mathsf{C}}(\ell_2) = \widehat{v}_2$$
$$\widehat{\rho}_1 = \widehat{\rho}_2 = \widehat{\rho}$$
$$(\widehat{\mathsf{C}}, \widehat{\rho}) \models (e_\star\ \mathsf{c}^{\ell_1})^{\ell_2}$$

for some place holder constant c

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Correctness Relations

$$R : V \times L \to \{\mathit{true}, \mathit{false}\}$$

Idea: $v \; R \; l$ means that the value v is described by the property l.

Correctness criterion: R is preserved under computation:

$$\begin{array}{ccccc} p & \vdash & v_1 & \leadsto & v_2 \\ & & \vdots & & \vdots \\ & & R & \Rightarrow & R \\ & & \vdots & & \vdots \\ p & \vdash & l_1 & \triangleright & l_2 \end{array}$$

logical relation:

$$(p \vdash \cdot \leadsto \cdot) \; (R \twoheadrightarrow R) \; (p \vdash \cdot \triangleright \cdot)$$

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Admissible Correctness Relations

$$v \; R \; l_1 \;\wedge\; l_1 \sqsubseteq l_2 \;\Rightarrow\; v \; R \; l_2$$

$$(\forall l \in L' \subseteq L : v \; R \; l) \;\Rightarrow\; v \; R \; (\bigsqcap L')$$ ($\{l \mid v \; R \; l\}$ is a Moore family)

Two consequences:

$$v \; R \; \top$$

$$v \; R \; l_1 \;\wedge\; v \; R \; l_2 \;\Rightarrow\; v \; R \; (l_1 \sqcap l_2)$$

Assumption: $(L, \sqsubseteq)$ is a complete lattice.

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Example: Data Flow Analysis

Correctness relation

$$R_{\mathsf{CP}} : \mathbf{State} \times \widehat{\mathbf{State}}_{\mathsf{CP}} \to \{\mathit{true}, \mathit{false}\}$$

is defined by

$$\sigma \; R_{\mathsf{CP}} \; \widehat{\sigma} \quad \text{iff} \quad \forall x \in \mathit{FV}(S_\star) : (\widehat{\sigma}(x) = \top \;\vee\; \sigma(x) = \widehat{\sigma}(x))$$

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Example: Control Flow Analysis

Correctness relation

$$R_{\mathsf{CFA}} : \mathbf{Val} \times (\widehat{\mathbf{Env}} \times \widehat{\mathbf{Val}}) \to \{\mathit{true}, \mathit{false}\}$$

is defined by

$$v \; R_{\mathsf{CFA}} \; (\widehat{\rho}, \widehat{v}) \quad \text{iff} \quad v \; \mathcal{V} \; (\widehat{\rho}, \widehat{v})$$

where $\mathcal{V}$ is given by:

$$v \; \mathcal{V} \; (\widehat{\rho}, \widehat{v}) \text{ iff } \begin{cases} \mathit{true} & \text{if } v = c \\ t \in \widehat{v} \wedge \forall x \in \mathit{dom}(\rho) : \rho(x) \; \mathcal{V} \; (\widehat{\rho}, \widehat{\rho}(x)) & \text{if } v = \texttt{close } t \texttt{ in } \rho \end{cases}$$

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Representation Functions

$$\beta : V \to L$$

Idea: β maps a value to the *best* property describing it.

Correctness criterion:

$$\begin{array}{ccccc}
p & \vdash & v_1 & \leadsto & v_2 \\
 & \beta & \downarrow & & \downarrow \;\beta \\
 & & & \Rightarrow & \\
 & & \sqsubseteq & & \sqsubseteq \\
p & \vdash & l_1 & \triangleright & l_2
\end{array}$$

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Equivalence of Correctness Criteria

Given a representation function β we define a correctness relation R_β by

$$v \; R_\beta \; l \quad \text{iff} \quad \beta(v) \sqsubseteq l$$

Given a correctness relation R we define a representation function β_R by

$$\beta_R(v) \; = \; \bigsqcap \{l \mid v \; R \; l\}$$

Lemma:

(i) Given $\beta : V \to L$, then the relation $R_\beta : V \times L \to \{\mathit{true}, \mathit{false}\}$ is an admissible correctness relation such that $\beta_{R_\beta} = \beta$.

(ii) Given an admissible correctness relation $R : V \times L \to \{\mathit{true}, \mathit{false}\}$, then β_R is well-defined and $R_{\beta_R} = R$.

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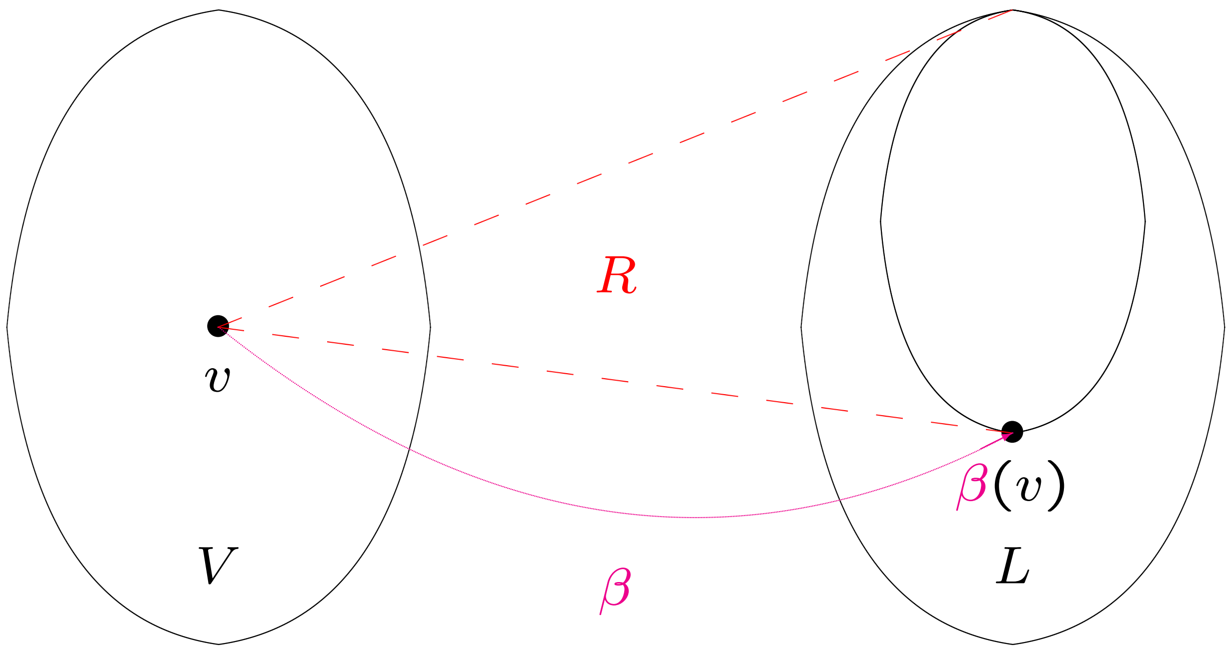
Equivalence of Criteria: R is *generated* by β



Example: Data Flow Analysis

Representation function

$$\beta_{\mathsf{CP}} : \mathbf{State} \to \widehat{\mathbf{State}}_{\mathsf{CP}}$$

is defined by

$$\beta_{\mathsf{CP}}(\sigma) = \lambda x.\sigma(x)$$

R_{CP} is *generated by* β_{CP}:

$$\sigma \; R_{\mathsf{CP}} \; \widehat{\sigma} \quad \underline{\text{iff}} \quad \beta_{\mathsf{CP}}(\sigma) \sqsubseteq_{\mathsf{CP}} \widehat{\sigma}$$

Example: Control Flow Analysis

Representation function

$$\beta_{\mathsf{CFA}} : \mathbf{Val} \to \widehat{\mathbf{Env}} \times \widehat{\mathbf{Val}}$$

is defined by

$$\beta_{\mathsf{CFA}}(v) = \begin{cases} (\lambda x.\emptyset, \emptyset) & \text{if } v = c \\ (\beta^{E}_{\mathsf{CFA}}(\rho), \{t\}) & \text{if } v = \texttt{close}\ t\ \texttt{in}\ \rho \end{cases}$$

$$\begin{aligned} \beta^{E}_{\mathsf{CFA}}(\rho)(x) \;=\; & \bigcup \{\widehat{\rho}_y(x) \mid \beta_{\mathsf{CFA}}(\rho(y)) = (\widehat{\rho}_y, \widehat{v}_y) \text{ and } y \in \mathit{dom}(\rho)\} \\ & \cup \begin{cases} \{\widehat{v}_x\} & \text{if } x \in \mathit{dom}(\rho) \text{ and } \beta_{\mathsf{CFA}}(\rho(x)) = (\widehat{\rho}_x, \widehat{v}_x) \\ \emptyset & \text{otherwise} \end{cases} \end{aligned}$$

R_{CFA} is *generated by* β_{CFA}:

$$v \; R_{\mathsf{CFA}} \; (\widehat{\rho}, \widehat{v}) \quad \underline{\text{iff}} \quad \beta_{\mathsf{CFA}}(v) \sqsubseteq_{\mathsf{CFA}} (\widehat{\rho}, \widehat{v})$$

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A Modest Generalisation

Semantics:

$$p \vdash v_1 \rightsquigarrow v_2$$

where $v_1 \in V_1, v_2 \in V_2$

Program analysis:

$$p \vdash l_1 \rhd l_2$$

where $l_1 \in L_1, l_2 \in L_2$

$$\begin{array}{ccccc} p & \vdash & v_1 & \rightsquigarrow & v_2 \\ & & \vdots & & \vdots \\ & & R_1 & \Rightarrow & R_2 \\ & & \vdots & & \vdots \\ p & \vdash & l_1 & \rhd & l_2 \end{array}$$

logical relation:

$$(p \vdash \cdot \rightsquigarrow \cdot)\ (R_1 \twoheadrightarrow R_2)\ (p \vdash \cdot \rhd \cdot)$$

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Higher-Order Formulation

Assume that

- R_1 is an admissible correctness relation for V_1 and L_1 that is *generated by* the representation function $\beta_1 : V_1 \rightarrow L_1$
- R_2 is an admissible correctness relation for V_2 and L_2 that is *generated by* the representation function $\beta_2 : V_2 \rightarrow L_2$

Then the relation $R_1 \twoheadrightarrow R_2$ is an admissible correctness relation for $V_1 \rightarrow V_2$ and $L_1 \rightarrow L_2$
that is *generated by* the representation function $\beta_1 \twoheadrightarrow \beta_2$ defined by

$$(\beta_1 \twoheadrightarrow \beta_2)(\leadsto) = \lambda l_1. \bigsqcup \{\beta_2(v_2) \mid \beta_1(v_1) \sqsubseteq l_1 \ \wedge \ v_1 \leadsto v_2\}$$

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Example:

Semantics:

$\texttt{plus} \vdash (z_1, z_2) \leadsto z_1 + z_2$

where $z_1, z_2 \in \mathbf{Z}$

Program analysis:

$\texttt{plus} \vdash ZZ \rhd \{z_1 + z_2 \mid (z_1, z_2) \in ZZ\}$

where $ZZ \subseteq \mathbf{Z} \times \mathbf{Z}$

	Correctness relations	Representation functions
result	R_{Z}	$\beta_{\mathsf{Z}}(z) = \{z\}$
argument	$R_{\mathsf{Z}\times\mathsf{Z}}$	$\beta_{\mathsf{Z}\times\mathsf{Z}}(z_1, z_2) = \{(z_1, z_2)\}$
$\texttt{plus}$	$(\texttt{plus} \vdash \cdot \leadsto \cdot)$ $(R_{\mathsf{Z}\times\mathsf{Z}} \twoheadrightarrow R_{\mathsf{Z}})$ $(\texttt{plus} \vdash \cdot \rhd \cdot)$	$(\beta_{\mathsf{Z}\times\mathsf{Z}} \twoheadrightarrow \beta_{\mathsf{Z}})(\texttt{plus} \vdash \cdot \leadsto \cdot) \sqsubseteq (\texttt{plus} \vdash \cdot \rhd \cdot)$

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Approximation of Fixed Points

- Fixed points

- Widening

- Narrowing

Example: lattice of intervals for *Array Bound Analysis*

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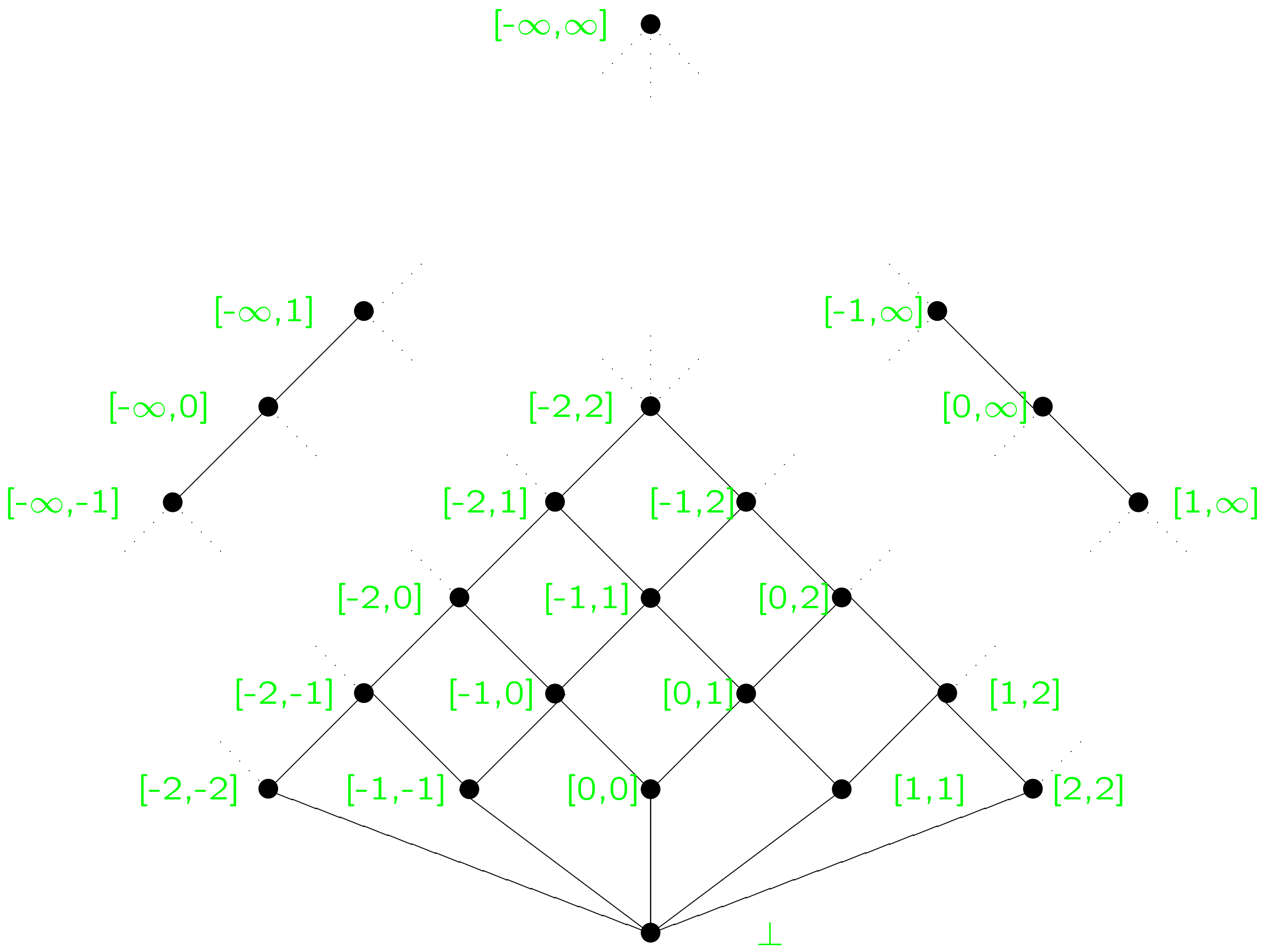
The complete lattice **Interval** = (**Interval**, $\sqsubseteq$)



Fixed points

Let $f : L \to L$ be a *monotone function* on a complete lattice $L = (L, \sqsubseteq, \sqcup, \sqcap, \bot, \top)$.

l is a *fixed point*	iff	$f(l) = l$	$\mathit{Fix}(f) = \{l \mid f(l) = l\}$
f is *reductive* at l	iff	$f(l) \sqsubseteq l$	$\mathit{Red}(f) = \{l \mid f(l) \sqsubseteq l\}$
f is *extensive* at l	iff	$f(l) \sqsupseteq l$	$\mathit{Ext}(f) = \{l \mid f(l) \sqsupseteq l\}$

Tarski's Theorem ensures that

$$\mathit{lfp}(f) = \bigsqcap \mathit{Fix}(f) = \bigsqcap \mathit{Red}(f) \in \mathit{Fix}(f) \subseteq \mathit{Red}(f)$$

$$\mathit{gfp}(f) = \bigsqcup \mathit{Fix}(f) = \bigsqcup \mathit{Ext}(f) \in \mathit{Fix}(f) \subseteq \mathit{Ext}(f)$$

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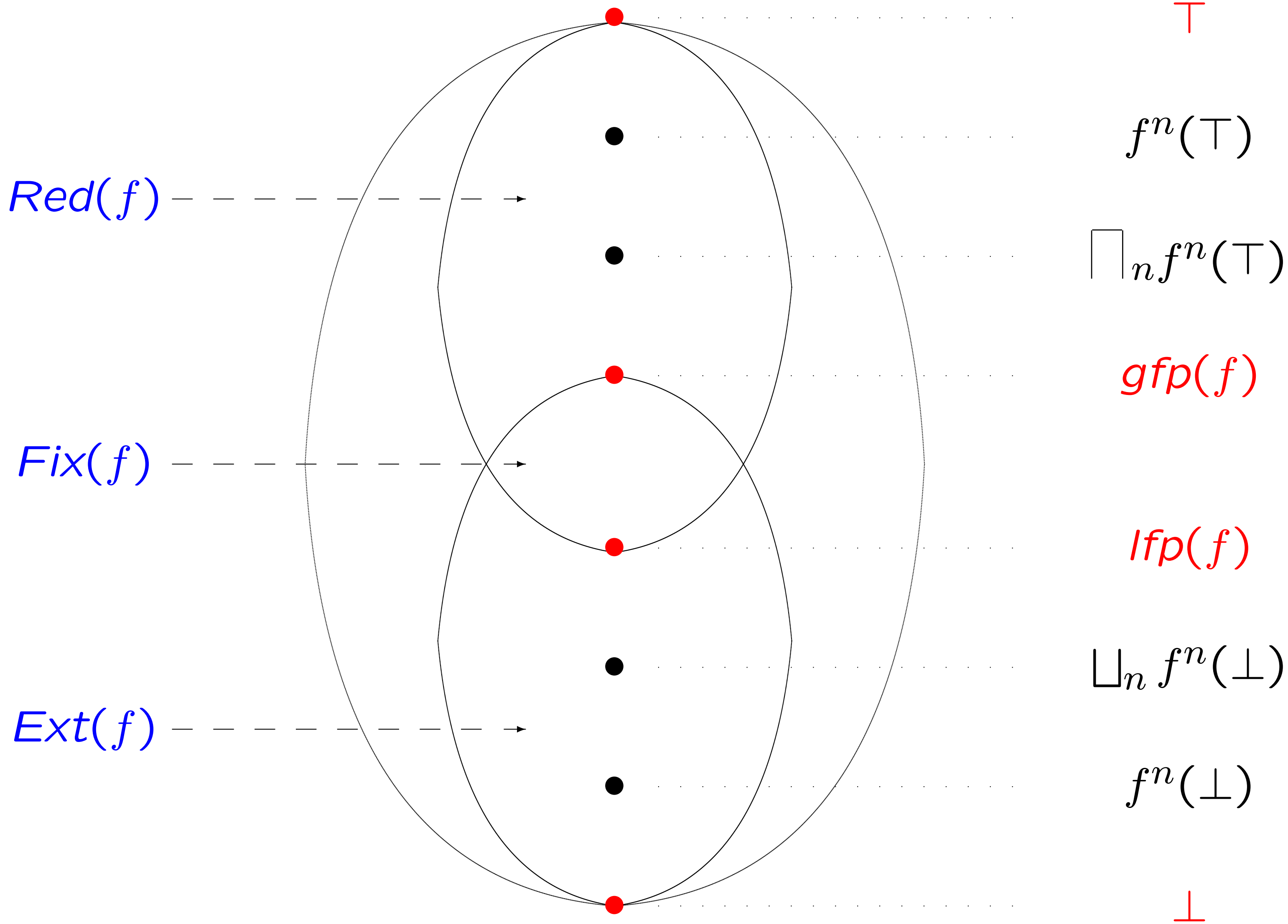
Fixed points of f



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Widening Operators

Problem: We cannot guarantee that $(f^n(\bot))_n$ eventually stabilises nor that its least upper bound necessarily equals *lfp*(f).

Idea: We replace $(f^n(\bot))_n$ by a new sequence $(f^n_\nabla)_n$ that is known to eventually stabilise and to do so with a value that is a safe (upper) approximation of the least fixed point.

The new sequence is parameterised on the widening operator ∇: an upper bound operator satisfying a finiteness condition.

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Upper bound operators

$\breve{\sqcup} : L \times L \to L$ is an *upper bound operator* iff

$$l_1 \ \sqsubseteq \ l_1 \ \breve{\sqcup} \ l_2 \ \sqsupseteq \ l_2$$

for all $l_1, l_2 \in L$.

Let $(l_n)_n$ be a sequence of elements of L. Define the sequence $(l_n^{\breve{\sqcup}})_n$ by:

$$l_n^{\breve{\sqcup}} = \begin{cases} l_n & \text{if } n = 0 \\ l_{n-1}^{\breve{\sqcup}} \ \breve{\sqcup} \ l_n & \text{if } n > 0 \end{cases}$$

Fact: If $(l_n)_n$ is a sequence and $\breve{\sqcup}$ is an upper bound operator then $(l_n^{\breve{\sqcup}})_n$ is an ascending chain; furthermore $l_n^{\breve{\sqcup}} \sqsupseteq \bigsqcup\{l_0, l_1, \cdots, l_n\}$ for all n.

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Example:

Let *int* be an arbitrary but fixed element of **Interval**.

An upper bound operator:

$$int_1 \mathbin{\check{\sqcup}}^{int} int_2 = \begin{cases} int_1 \sqcup int_2 & \text{if } int_1 \sqsubseteq int \ \vee\ int_2 \sqsubseteq int_1 \\ [-\infty, \infty] & \text{otherwise} \end{cases}$$

Example: $[1,2] \mathbin{\check{\sqcup}}^{[0,2]} [2,3] = [1,3]$ and $[2,3] \mathbin{\check{\sqcup}}^{[0,2]} [1,2] = [-\infty, \infty]$.

Transformation of:	$[0,0], [1,1], [2,2], [3,3], [4,4], [5,5], \cdots$
If $int = [0,\infty]$:	$[0,0], [0,1], [0,2], [0,3], [0,4], [0,5], \cdots$
If $int = [0,2]$:	$[0,0], [0,1], [0,2], [0,3], [-\infty,\infty], [-\infty,\infty], \cdots$

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Widening operators

An operator $\nabla : L \times L \to L$ is a *widening operator* iff

- it is an upper bound operator, and
- for all ascending chains $(l_n)_n$ the ascending chain $(l_n^{\nabla})_n$ eventually stabilises.

Widening operators

Given a monotone function $f : L \to L$ and a widening operator ∇ define the sequence $(f_\nabla^n)_n$ by

$$f_\nabla^n = \begin{cases} \bot & \text{if } n = 0 \\ f_\nabla^{n-1} & \text{if } n > 0 \ \wedge \ f(f_\nabla^{n-1}) \sqsubseteq f_\nabla^{n-1} \\ f_\nabla^{n-1} \ \nabla \ f(f_\nabla^{n-1}) & \text{otherwise} \end{cases}$$

One can show that:

- $(f_\nabla^n)_n$ is an ascending chain that eventually stabilises
- it happens when $f(f_\nabla^m) \sqsubseteq f_\nabla^m$ for some value of m
- Tarski's Theorem then gives $f_\nabla^m \sqsupseteq \mathit{lfp}(f)$

$\mathit{lfp}_\nabla(f) = f_\nabla^m$

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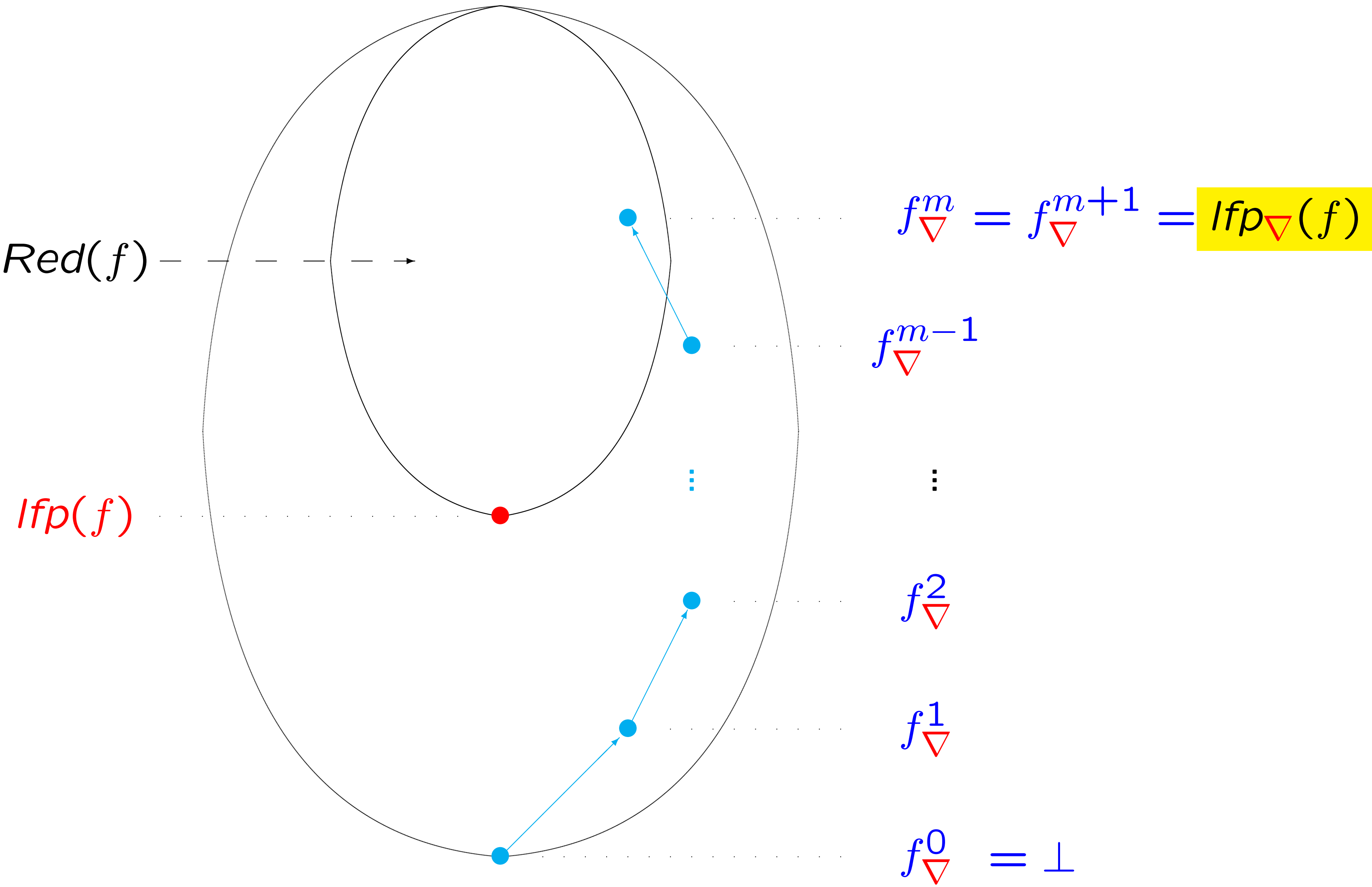
The widening operator ∇ applied to f



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Example:

Let K be a *finite* set of integers, e.g. the set of integers explicitly mentioned in a given program.

We shall define a widening operator ∇ based on K.

Idea: $[z_1, z_2] \ \nabla \ [z_3, z_4]$ is

$$[\ \mathsf{LB}(z_1, z_3)\ ,\ \mathsf{UB}(z_2, z_4)\ ]$$

where

- $\mathsf{LB}(z_1, z_3) \in \{z_1\} \cup K \cup \{-\infty\}$ is the best possible lower bound, and
- $\mathsf{UB}(z_2, z_4) \in \{z_2\} \cup K \cup \{\infty\}$ is the best possible upper bound.

The effect: a change in any of the bounds of the interval $[z_1, z_2]$ can only take place finitely many times – corresponding to the cardinality of K.

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Example (cont.) — formalisation:

Let $z_i \in \mathbf{Z}' = \mathbf{Z} \cup \{-\infty, \infty\}$ and write:

$$\mathsf{LB}_K(z_1, z_3) = \begin{cases} z_1 & \text{if } z_1 \leq z_3 \\ k & \text{if } z_3 < z_1 \ \wedge \ k = \max\{k \in K \mid k \leq z_3\} \\ -\infty & \text{if } z_3 < z_1 \ \wedge \ \forall k \in K : z_3 < k \end{cases}$$

$$\mathsf{UB}_K(z_2, z_4) = \begin{cases} z_2 & \text{if } z_4 \leq z_2 \\ k & \text{if } z_2 < z_4 \ \wedge \ k = \min\{k \in K \mid z_4 \leq k\} \\ \infty & \text{if } z_2 < z_4 \ \wedge \ \forall k \in K : k < z_4 \end{cases}$$

$$int_1 \ \nabla \ int_2 = \begin{cases} \bot & \text{if } int_1 = int_2 = \bot \\ [\ \mathsf{LB}_K(\inf(int_1), \inf(int_2))\ ,\ \mathsf{UB}_K(\sup(int_1), \sup(int_2))\ ] & \text{otherwise} \end{cases}$$

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Example (cont.):

Consider the ascending chain $(int_n)_n$

$$[0,1],[0,2],[0,3],[0,4],[0,5],[0,6],[0,7],\cdots$$

and assume that $K = \{3,5\}$.

Then $(int_n^{\nabla})_n$ is the chain

$$[0,1],[0,3],[0,3],[0,5],[0,5],[0,\infty],[0,\infty],\cdots$$

which eventually stabilises.

Narrowing Operators

Status: Widening gives us an upper approximation $\mathit{lfp}_\nabla(f)$ of the least fixed point of f.

Observation: $f(\mathit{lfp}_\nabla(f)) \sqsubseteq \mathit{lfp}_\nabla(f)$ so the approximation can be improved by considering the iterative sequence $(f^n(\mathit{lfp}_\nabla(f)))_n$.

It will satisfy $f^n(\mathit{lfp}_\nabla(f)) \sqsupseteq \mathit{lfp}(f)$ for all n so we can stop at an arbitrary point.

The notion of narrowing is *one way* of encapsulating a termination criterion for the sequence.

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Narrowing

An operator $\triangle : L \times L \to L$ is a *narrowing operator* iff

- $l_2 \sqsubseteq l_1 \;\Rightarrow\; l_2 \sqsubseteq (l_1 \triangle l_2) \sqsubseteq l_1$ for all $l_1, l_2 \in L$, and
- for all descending chains $(l_n)_n$ the sequence $(l_n^{\triangle})_n$ eventually stabilises.

Recall: The sequence $(l_n^{\triangle})_n$ is defined by:

$$l_n^{\triangle} = \begin{cases} l_n & \text{if } n = 0 \\ l_{n-1}^{\triangle} \triangle l_n & \text{if } n > 0 \end{cases}$$

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Narrowing

We construct the sequence $([f]^n_\triangle)_n$

$$[f]^n_\triangle = \begin{cases} \mathit{lfp}_\nabla(f) & \text{if } n = 0 \\ [f]^{n-1}_\triangle \;\triangle\; f([f]^{n-1}_\triangle) & \text{if } n > 0 \end{cases}$$

One can show that:

- $([f]^n_\triangle)_n$ is a descending chain where all elements satisfy $\mathit{lfp}(f) \sqsubseteq [f]^n_\triangle$

- the chain eventually stabilises so $[f]^{m'}_\triangle = [f]^{m'+1}_\triangle$ for some value m'

$$\mathit{lfp}^{\triangle}_{\nabla}(f) = [f]^{m'}_\triangle$$

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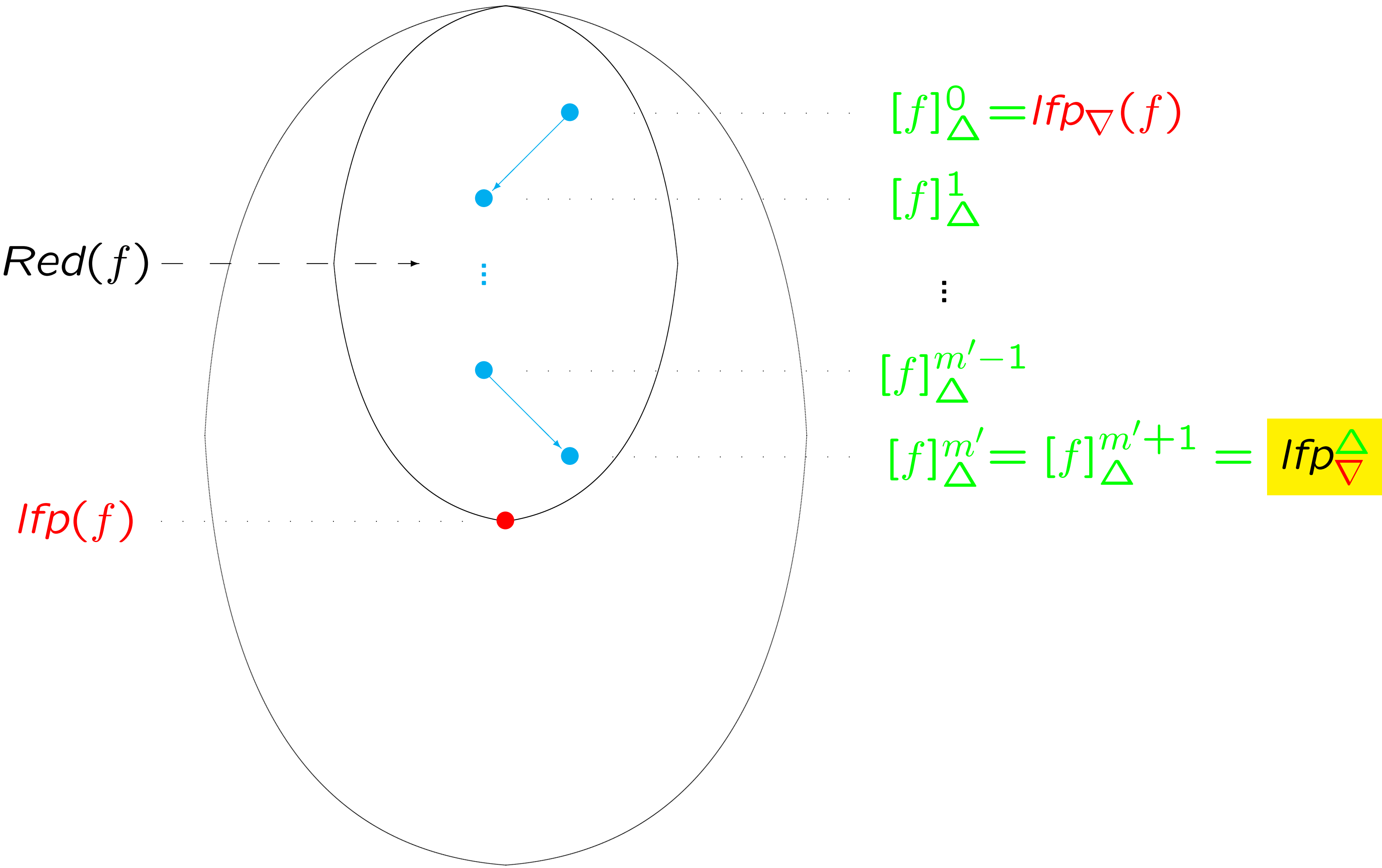
The narrowing operator Δ applied to f



Example:

The complete lattice $(\mathbf{Interval}, \sqsubseteq)$ has two kinds of infinite descending chains:

- those with elements of the form $[-\infty, z]$, $z \in \mathbf{Z}$
- those with elements of the form $[z, \infty]$, $z \in \mathbf{Z}$

Idea: Given some fixed non-negative number N the narrowing operator $\triangle_N$ will force an infinite descending chain

$$[z_1, \infty], [z_2, \infty], [z_3, \infty], \cdots$$

(where $z_1 < z_2 < z_3 < \cdots$) to stabilise when $z_i > N$

Similarly, for a descending chain with elements of the form $[-\infty, z_i]$ the narrowing operator will force it to stabilise when $z_i < -N$

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Example (cont.) — formalisation:

Define $\triangle = \triangle_N$ by

$$int_1 \triangle int_2 = \begin{cases} \bot & \text{if } int_1 = \bot \lor int_2 = \bot \\ [z_1, z_2] & \text{otherwise} \end{cases}$$

where

$$z_1 = \begin{cases} \inf(int_1) & \text{if } N < \inf(int_2) \land \sup(int_2) = \infty \\ \inf(int_2) & \text{otherwise} \end{cases}$$

$$z_2 = \begin{cases} \sup(int_1) & \text{if } \inf(int_2) = -\infty \land \sup(int_2) < -N \\ \sup(int_2) & \text{otherwise} \end{cases}$$

Example (cont.):

Consider the infinite descending chain $([n,\infty])_n$

$$[0,\infty],[1,\infty],[2,\infty],[3,\infty],[4,\infty],[5,\infty],\cdots$$

and assume that $N = 3$.

Then the narrowing operator $\triangle_N$ will give the sequence $([n,\infty]^{\triangle})_n$

$$[0,\infty],[1,\infty],[2,\infty],[3,\infty],[3,\infty],[3,\infty],\cdots$$

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Galois Connections

- Galois connections and adjunctions
- Extraction functions
- Galois insertions
- Reduction operators

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Galois connections

$$L \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} M$$

α: *abstraction function*

γ: *concretisation function*

is a Galois connection if and only if

α and γ are monotone functions

that satisfy

$$\gamma \circ \alpha \sqsupseteq \lambda l.l$$

$$\alpha \circ \gamma \sqsubseteq \lambda m.m$$

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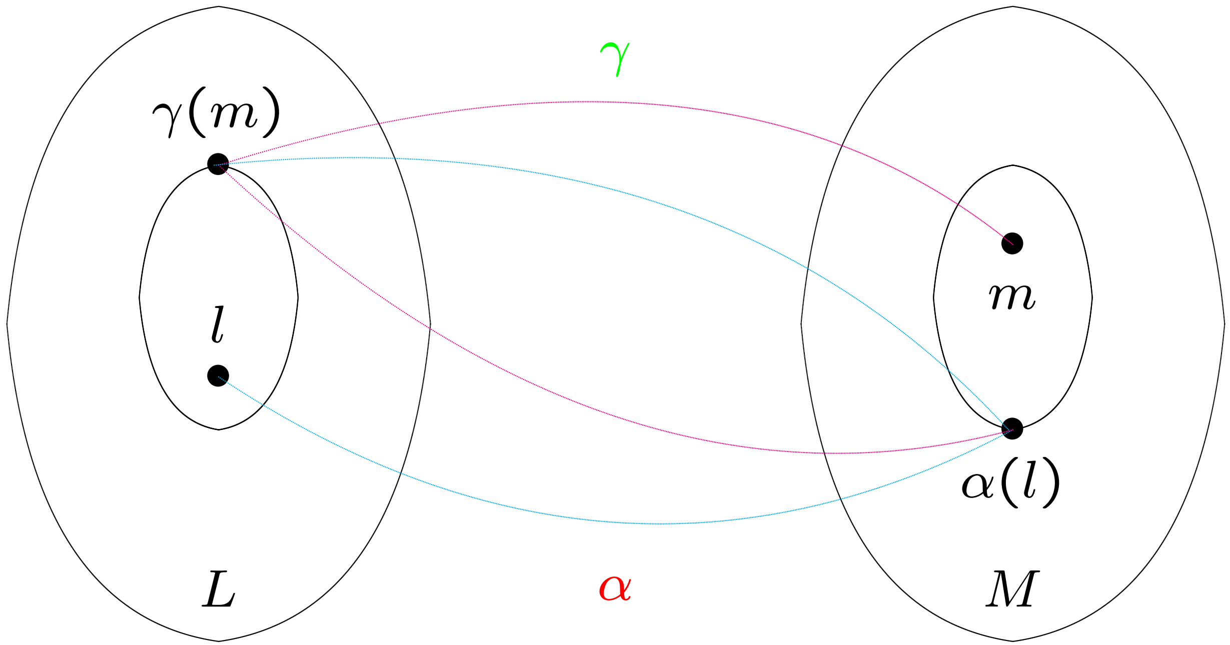
Galois connections



$\gamma \circ \alpha \sqsupseteq \lambda l.l$ $\qquad$ $\alpha \circ \gamma \sqsubseteq \lambda m.m$

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Example:

Galois connection

$$(\mathcal{P}(\mathbf{Z}), \alpha_{\mathrm{ZI}}, \gamma_{\mathrm{ZI}}, \textbf{Interval})$$

with concretisation function

$$\gamma_{\mathrm{ZI}}(\mathit{int}) = \{z \in \mathbf{Z} \mid \mathsf{inf}(\mathit{int}) \leq z \leq \mathsf{sup}(\mathit{int})\}$$

and abstraction function

$$\alpha_{\mathrm{ZI}}(Z) = \begin{cases} \bot & \text{if } Z = \emptyset \\ [\mathsf{inf}'(Z), \mathsf{sup}'(Z)] & \text{otherwise} \end{cases}$$

Examples:

$$\begin{aligned} \gamma_{\mathrm{ZI}}([0,3]) &= \{0,1,2,3\} \\ \gamma_{\mathrm{ZI}}([0,\infty]) &= \{z \in \mathbf{Z} \mid z \geq 0\} \\ \alpha_{\mathrm{ZI}}(\{0,1,3\}) &= [0,3] \\ \alpha_{\mathrm{ZI}}(\{2 * z \mid z > 0\}) &= [2,\infty] \end{aligned}$$

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Adjunctions

$$L \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} M$$

is an *adjunction* if and only if

$$\alpha : L \to M \text{ and } \gamma : M \to L \text{ are total functions}$$

that satisfy

$$\alpha(l) \sqsubseteq m \qquad \underline{\text{iff}} \qquad l \sqsubseteq \gamma(m)$$

for all $l \in L$ and $m \in M$.

Proposition: (α, γ) is an adjunction iff it is a Galois connection.

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Galois connections from representation functions

A representation function $\beta : V \to L$ gives rise to a Galois connection

$$(\mathcal{P}(V), \alpha, \gamma, L)$$

where

$$\begin{aligned} \alpha(V') &= \bigsqcup\{\beta(v) \mid v \in V'\} \\ \gamma(l) &= \{v \in V \mid \beta(v) \sqsubseteq l\} \end{aligned}$$

for $V' \subseteq V$ and $l \in L$.

This indeed defines an adjunction:

$$\begin{aligned} \alpha(V') \sqsubseteq l &\Leftrightarrow \bigsqcup\{\beta(v) \mid v \in V'\} \sqsubseteq l \\ &\Leftrightarrow \forall v \in V' : \beta(v) \sqsubseteq l \\ &\Leftrightarrow V' \subseteq \gamma(l) \end{aligned}$$

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Galois connections from extraction functions

An *extraction function*

$$\eta : V \rightarrow D$$

maps the values of V to their best descriptions in D.

It gives rise to a representation function $\beta_\eta : V \rightarrow \mathcal{P}(D)$ (corresponding to $L = (\mathcal{P}(D), \subseteq)$) defined by

$$\beta_\eta(v) = \{\eta(v)\}$$

The associated Galois connection is

$$(\mathcal{P}(V), \alpha_\eta, \gamma_\eta, \mathcal{P}(D))$$

where

$$\begin{array}{rcl} \alpha_\eta(V') &=& \bigcup\{\beta_\eta(v) \mid v \in V'\} \quad = \quad \{\eta(v) \mid v \in V'\} \\ \gamma_\eta(D') &=& \{v \in V \mid \beta_\eta(v) \subseteq D'\} \quad = \quad \{v \mid \eta(v) \in D'\} \end{array}$$

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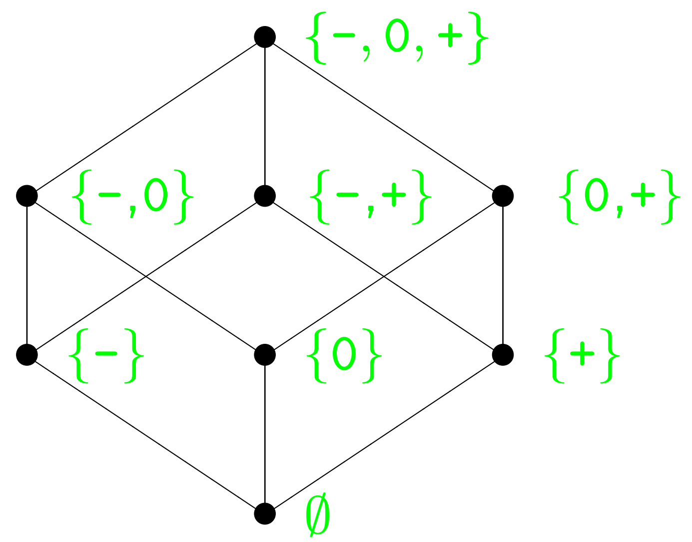
Example:

Extraction function

$$\text{sign} : \mathbf{Z} \to \mathbf{Sign}$$

specified by

$$\text{sign}(z) = \begin{cases} - & \text{if } z < 0 \\ 0 & \text{if } z = 0 \\ + & \text{if } z > 0 \end{cases}$$



Galois connection

$$(\mathcal{P}(\mathbf{Z}), \alpha_{\text{sign}}, \gamma_{\text{sign}}, \mathcal{P}(\mathbf{Sign}))$$

with

$$\begin{aligned} \alpha_{\text{sign}}(Z) &= \{\text{sign}(z) \mid z \in Z\} \\ \gamma_{\text{sign}}(S) &= \{z \in \mathbf{Z} \mid \text{sign}(z) \in S\} \end{aligned}$$

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Properties of Galois Connections

Lemma: If (L, α, γ, M) is a Galois connection then:

- α uniquely determines γ by $\gamma(m) = \bigsqcup\{l \mid \alpha(l) \sqsubseteq m\}$
- γ uniquely determines α by $\alpha(l) = \bigsqcap\{m \mid l \sqsubseteq \gamma(m)\}$
- α is completely additive and γ is completely multiplicative

In particular $\alpha(\bot) = \bot$ and $\gamma(\top) = \top$.

Lemma:

- If $\alpha : L \to M$ is completely additive then there exists (an upper adjoint) $\gamma : M \to L$ such that (L, α, γ, M) is a Galois connection.
- If $\gamma : M \to L$ is completely multiplicative then there exists (a lower adjoint) $\alpha : L \to M$ such that (L, α, γ, M) is a Galois connection.

Fact: If (L, α, γ, M) is a Galois connection then

- $\alpha \circ \gamma \circ \alpha = \alpha$ and $\gamma \circ \alpha \circ \gamma = \gamma$

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Example:

Define $\gamma_{\text{IS}} : \mathcal{P}(\mathbf{Sign}) \to \textbf{Interval}$ by:

$$\begin{array}{rcl} \gamma_{\text{IS}}(\{\text{-},0,\text{+}\}) & = & [-\infty,\infty] \\ \gamma_{\text{IS}}(\{\text{-},\text{+}\}) & = & [-\infty,\infty] \\ \gamma_{\text{IS}}(\{\text{-}\}) & = & [-\infty,-1] \\ \gamma_{\text{IS}}(\{\text{+}\}) & = & [1,\infty] \end{array} \qquad \begin{array}{rcl} \gamma_{\text{IS}}(\{\text{-},0\}) & = & [-\infty,0] \\ \gamma_{\text{IS}}(\{0,\text{+}\}) & = & [0,\infty] \\ \gamma_{\text{IS}}(\{0\}) & = & [0,0] \\ \gamma_{\text{IS}}(\emptyset) & = & \bot \end{array}$$

Does there exist an abstraction function

$$\alpha_{\text{IS}} : \textbf{Interval} \to \mathcal{P}(\mathbf{Sign})$$

such that $(\textbf{Interval}, \alpha_{\text{IS}}, \gamma_{\text{IS}}, \mathcal{P}(\mathbf{Sign}))$ is a Galois connection?

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Example (cont.):

Is γ_{IS} completely multiplicative?

- if yes: then there exists a Galois connection
- if no: then there cannot exist a Galois connection

Lemma: If L and M are complete lattices and M is finite then $\gamma : M \to L$ is completely multiplicative if and only if the following hold:

- $\gamma : M \to L$ is monotone,
- $\gamma(\top) = \top$, and
- $\gamma(m_1 \sqcap m_2) = \gamma(m_1) \sqcap \gamma(m_2)$ whenever $m_1 \not\sqsubseteq m_2 \wedge m_2 \not\sqsubseteq m_1$

We calculate

$$\begin{array}{rcccl} \gamma_{\text{IS}}(\{\text{-},0\} \cap \{\text{-},\text{+}\}) & = & \gamma_{\text{IS}}(\{\text{-}\}) & = & [-\infty,-1] \\ \gamma_{\text{IS}}(\{\text{-},0\}) \sqcap \gamma_{\text{IS}}(\{\text{-},\text{+}\}) & = & [-\infty,0] \sqcap [-\infty,\infty] & = & [-\infty,0] \end{array}$$

showing that there is no Galois connection involving γ_{IS}.

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Galois Connections are the Right Concept

We use the mundane approach to correctness to demonstrate this for:

- Admissible correctness relations

- Representation functions

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The mundane approach: correctness relations

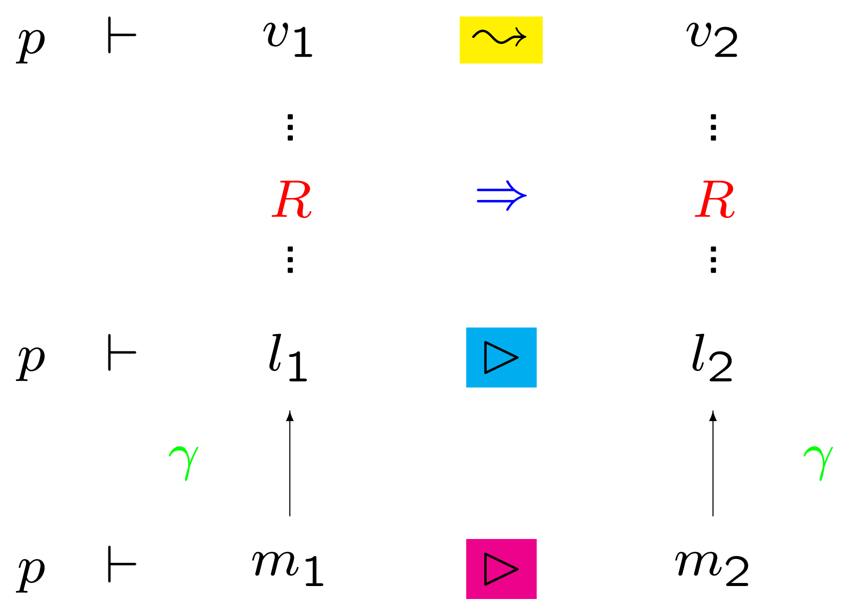
Assume

- $R : V \times L \to \{\textit{true}, \textit{false}\}$ is an admissible correctness relation
- (L, α, γ, M) is a Galois connection

Then $S : V \times M \to \{\textit{true}, \textit{false}\}$ defined by

$$v \; S \; m \quad \underline{\text{iff}} \quad v \; R \; (\gamma(m))$$

is an admissible correctness relation between V and M



The mundane approach: representation functions

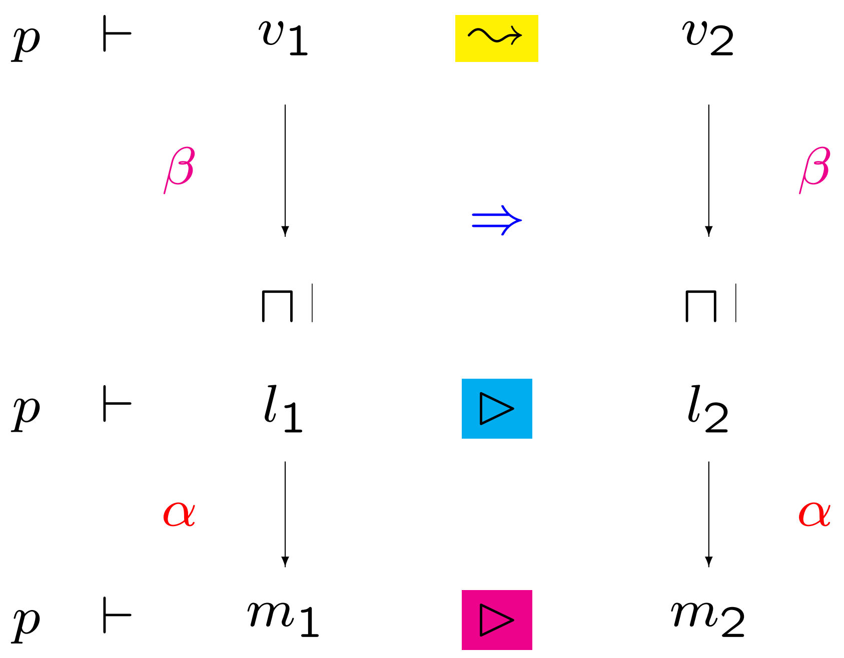
Assume

- $R : V \times L \to \{$*true*, *false*$\}$ is *generated by* $\beta : V \to L$
- (L, α, γ, M) is a Galois connection

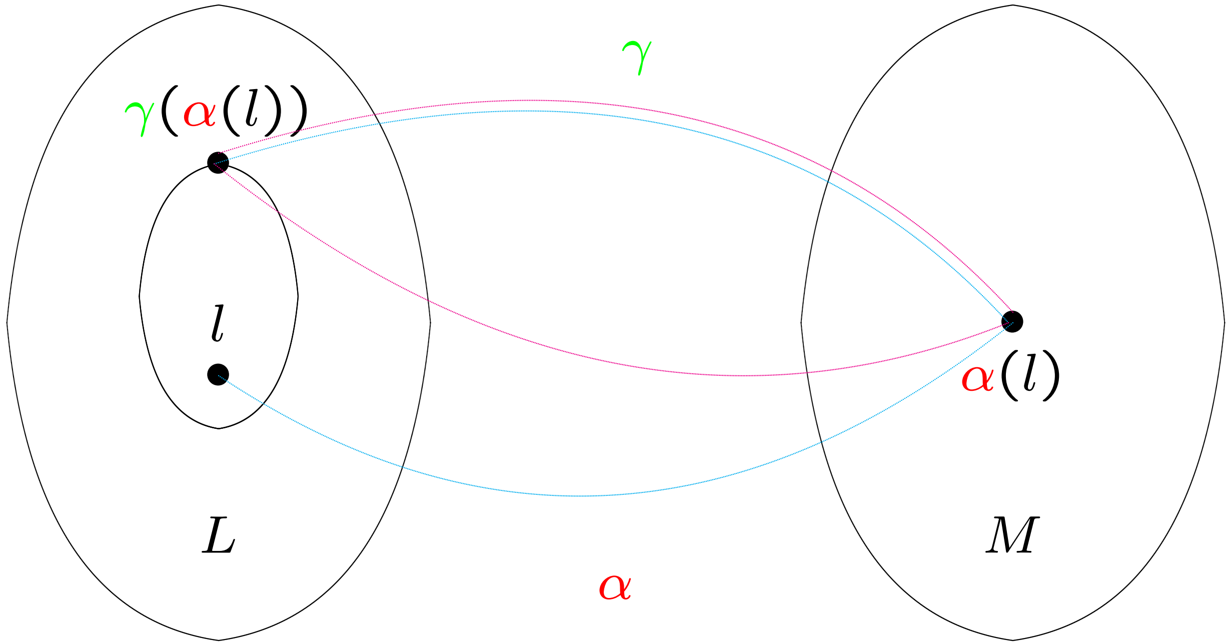
Then $S : V \times M \to \{$*true*, *false*$\}$ defined by

$$v \; S \; m \quad \text{\underline{iff}} \quad v \; R \; (\gamma(m))$$

is *generated by* $\alpha \circ \beta : V \to M$



Galois Insertions



Monotone functions satisfying: $\gamma \circ \alpha \sqsupseteq \lambda l.l \qquad \alpha \circ \gamma = \lambda m.m$

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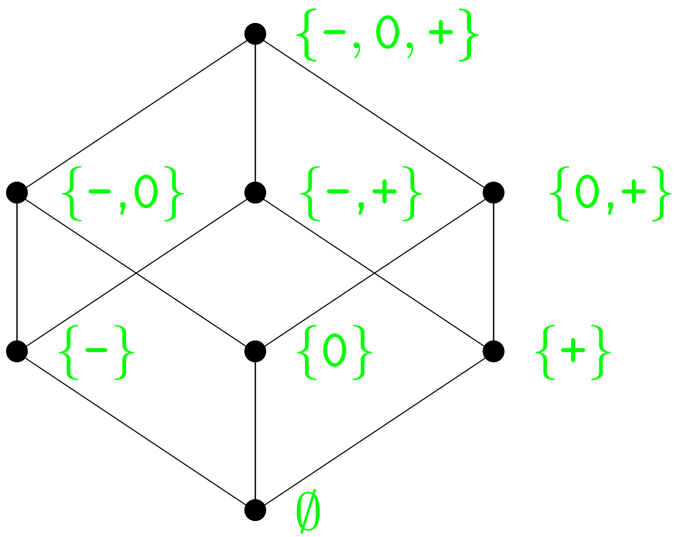
Example (1):

$$(\mathcal{P}(\mathbf{Z}), \alpha_{\text{sign}}, \gamma_{\text{sign}}, \mathcal{P}(\mathbf{Sign}))$$

where sign : $\mathbf{Z} \rightarrow \mathbf{Sign}$ is specified by:

$$\text{sign}(z) = \begin{cases} - & \text{if } z < 0 \\ 0 & \text{if } z = 0 \\ + & \text{if } z > 0 \end{cases}$$



Is it a Galois insertion?

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Example (2):

$$(\mathcal{P}(\mathbf{Z}), \alpha_{\text{signparity}}, \gamma_{\text{signparity}}, \mathcal{P}(\mathbf{Sign} \times \textbf{Parity}))$$

where $\mathbf{Sign} = \{\text{-}, 0, \text{+}\}$ and $\textbf{Parity} = \{\text{odd}, \text{even}\}$

and signparity : $\mathbf{Z} \to \mathbf{Sign} \times \textbf{Parity}$:

$$\text{signparity}(z) = \begin{cases} (\text{sign}(z), \text{odd}) & \text{if } z \text{ is odd} \\ (\text{sign}(z), \text{even}) & \text{if } z \text{ is even} \end{cases}$$

Is it a Galois insertion?

Properties of Galois Insertions

Lemma: For a Galois connection (L, α, γ, M) the following claims are equivalent:

(i) (L, α, γ, M) is a Galois insertion;

(ii) α is surjective: $\forall m \in M : \exists l \in L : \alpha(l) = m$;

(iii) γ is injective: $\forall m_1, m_2 \in M : \gamma(m_1) = \gamma(m_2) \Rightarrow m_1 = m_2$; and

(iv) γ is an order-similarity: $\forall m_1, m_2 \in M : \gamma(m_1) \sqsubseteq \gamma(m_2) \Leftrightarrow m_1 \sqsubseteq m_2$.

Corollary: A Galois connection specified by an *extraction* function $\eta : V \rightarrow D$ is a Galois insertion if and only if η is surjective.

Example (1) reconsidered:

$$(\mathcal{P}(\mathbf{Z}), \alpha_{\mathsf{sign}}, \gamma_{\mathsf{sign}}, \mathcal{P}(\mathbf{Sign}))$$

$$\mathsf{sign}(z) = \begin{cases} \texttt{-} & \text{if } z < 0 \\ \texttt{0} & \text{if } z = 0 \\ \texttt{+} & \text{if } z > 0 \end{cases}$$

is a Galois insertion because sign is surjective.

Example (2) reconsidered:

$$(\mathcal{P}(\mathbf{Z}), \alpha_{\mathsf{signparity}}, \gamma_{\mathsf{signparity}}, \mathcal{P}(\mathbf{Sign} \times \mathbf{Parity}))$$

$$\mathsf{signparity}(z) = \begin{cases} (\mathsf{sign}(z), \mathsf{odd}) & \text{if } z \text{ is odd} \\ (\mathsf{sign}(z), \mathsf{even}) & \text{if } z \text{ is even} \end{cases}$$

is not a Galois insertion because signparity is not surjective.

Reduction Operators

Given a Galois connection (L, α, γ, M) it is always possible to obtain a Galois insertion by enforcing that the concretisation function γ is injective.

Idea: remove the superfluous elements from M using a *reduction operator*

$$\varsigma : M \to M$$

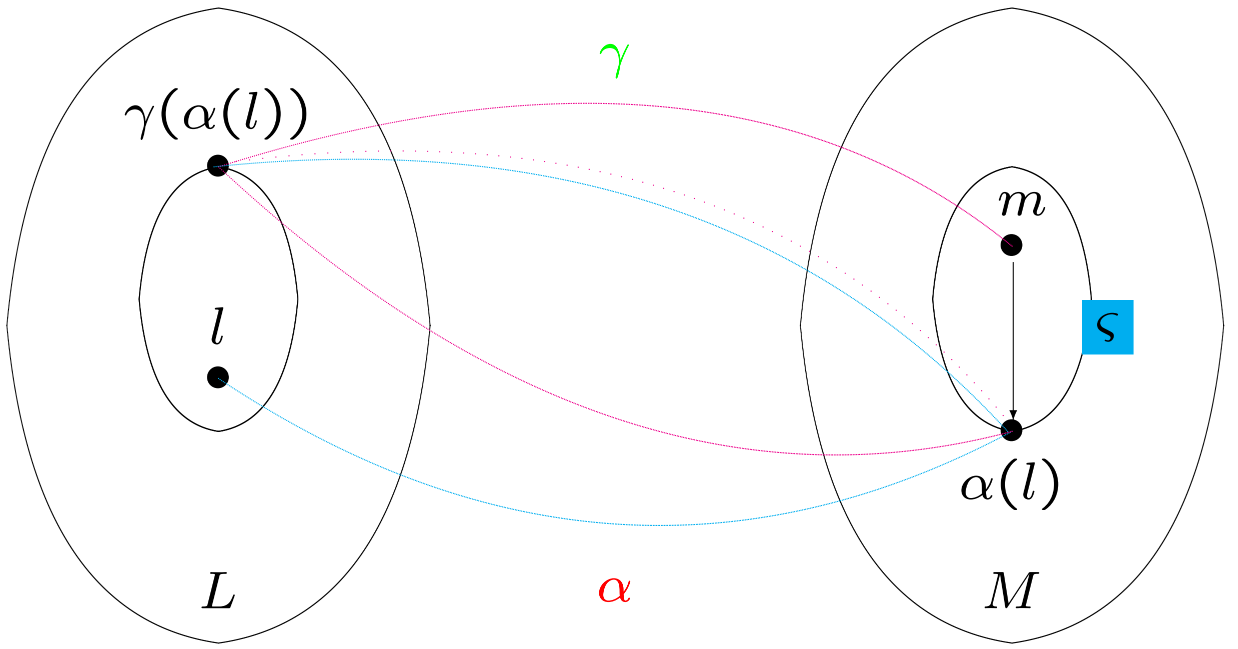
defined from the Galois connection.

Proposition: Let (L, α, γ, M) be a Galois connection and define the reduction operator $\varsigma : M \to M$ by

$$\varsigma(m) = \bigsqcap\{m' \mid \gamma(m) = \gamma(m')\}$$

Then $\varsigma[M] = (\{\varsigma(m) \mid m \in M\}, \sqsubseteq_M)$ is a complete lattice and $(L, \alpha, \gamma, \varsigma[M])$ is a Galois insertion.

The reduction operator $\varsigma : M \to M$



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Reduction operators from extraction functions

Assume that the Galois connection $(\mathcal{P}(V), \alpha_\eta, \gamma_\eta, \mathcal{P}(D))$ is given by an extraction function $\eta : V \to D$.

Then the reduction operator ς_η is given by

$$\varsigma_\eta(D') = D' \cap \eta[V]$$

where $\eta[V] = \{d \in D \mid \exists v \in V : \eta(v) = d\}$.

Since $\varsigma_\eta[\mathcal{P}(D)]$ is isomorphic to $\mathcal{P}(\eta[V])$ the resulting Galois insertion is isomorphic to

$$(\mathcal{P}(V), \alpha_\eta, \gamma_\eta, \mathcal{P}(\eta[V]))$$

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Systematic Design of Galois Connections

The "functional composition" (or "sequential composition") of two Galois connections is also a Galois connection:

$$L_0 \underset{\alpha_1}{\overset{\gamma_1}{\leftrightarrows}} L_1 \underset{\alpha_2}{\overset{\gamma_2}{\leftrightarrows}} L_2 \underset{\alpha_3}{\overset{\gamma_3}{\leftrightarrows}} \cdots \underset{\alpha_k}{\overset{\gamma_k}{\leftrightarrows}} L_k$$

A catalogue of techniques for combining Galois connections:

- independent attribute method
- direct product
- reduced product
- total function space

- relational method
- direct tensor product
- reduced tensor product
- monotone function space

Running Example: Array Bound Analysis

Approximation of the difference in magnitude between two numbers (typically the index and the bound):

- a Galois connection for approximating pairs (z_1, z_2) of integers by their difference $|z_1| - |z_2|$
- a Galois connection for approximating integers using a finite lattice $\{$<-1, -1, 0, +1, >+1$\}$
- a Galois connection for their functional composition

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Example: Difference in Magnitude

$$(\mathcal{P}(\mathbf{Z} \times \mathbf{Z}), \alpha_{\mathsf{diff}}, \gamma_{\mathsf{diff}}, \mathcal{P}(\mathbf{Z}))$$

where the extraction function diff : $\mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}$ calculates the difference in magnitude:

$$\mathsf{diff}(z_1, z_2) = |z_1| - |z_2|$$

The abstraction and concretisation functions are

$$\begin{aligned} \alpha_{\mathsf{diff}}(ZZ) &= \{|z_1| - |z_2| \mid (z_1, z_2) \in ZZ\} \\ \gamma_{\mathsf{diff}}(Z) &= \{(z_1, z_2) \mid |z_1| - |z_2| \in Z\} \end{aligned}$$

for $ZZ \subseteq \mathbf{Z} \times \mathbf{Z}$ and $Z \subseteq \mathbf{Z}$.

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Example: Finite Approximation

$$(\mathcal{P}(\mathbf{Z}), \alpha_{\mathsf{range}}, \gamma_{\mathsf{range}}, \mathcal{P}(\mathbf{Range}))$$

where $\mathbf{Range} = \{\texttt{<-1}, \texttt{-1}, \texttt{0}, \texttt{+1}, \texttt{>+1}\}$
and the extraction function $\mathsf{range} : \mathbf{Z} \to \mathbf{Range}$ is

$$\mathsf{range}(z) = \begin{cases} \texttt{<-1} & \text{if } z < -1 \\ \texttt{-1} & \text{if } z = -1 \\ \texttt{0} & \text{if } z = 0 \\ \texttt{+1} & \text{if } z = 1 \\ \texttt{>+1} & \text{if } z > 1 \end{cases}$$

The abstraction and concretisation functions are

$$\begin{aligned} \alpha_{\mathsf{range}}(Z) &= \{\mathsf{range}(z) \mid z \in Z\} \\ \gamma_{\mathsf{range}}(R) &= \{z \mid \mathsf{range}(z) \in R\} \end{aligned}$$

for $Z \subseteq \mathbf{Z}$ and $R \subseteq \mathbf{Range}$.

Example: Functional Composition

$$(\mathcal{P}(\mathbf{Z} \times \mathbf{Z}), \alpha_{\mathsf{R}}, \gamma_{\mathsf{R}}, \mathcal{P}(\textbf{Range}))$$

where

$$\begin{aligned} \alpha_{\mathsf{R}} &= \alpha_{\mathsf{range}} \circ \alpha_{\mathsf{diff}} \\ \gamma_{\mathsf{R}} &= \gamma_{\mathsf{diff}} \circ \gamma_{\mathsf{range}} \end{aligned}$$

The explicit formulae for the abstraction and concretisation functions

$$\begin{aligned} \alpha_{\mathsf{R}}(ZZ) &= \{\mathsf{range}(|z_1| - |z_2|) \mid (z_1, z_2) \in ZZ\} \\ \gamma_{\mathsf{R}}(R) &= \{(z_1, z_2) \mid \mathsf{range}(|z_1| - |z_2|) \in R\} \end{aligned}$$

correspond to the extraction function $\mathsf{range} \circ \mathsf{diff}$.

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Approximation of Pairs

Independent Attribute Method

Let $(L_1, \alpha_1, \gamma_1, M_1)$ and $(L_2, \alpha_2, \gamma_2, M_2)$ be Galois connections.

The *independent attribute method* gives a Galois connection

$$(L_1 \times L_2, \alpha, \gamma, M_1 \times M_2)$$

where

$$\begin{aligned}\alpha(l_1, l_2) &= (\alpha_1(l_1), \alpha_2(l_2)) \\ \gamma(m_1, m_2) &= (\gamma_1(m_1), \gamma_2(m_2))\end{aligned}$$

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Example: Detection of Signs Analysis

Given

$$(\mathcal{P}(\mathbf{Z}), \alpha_{\mathsf{sign}}, \gamma_{\mathsf{sign}}, \mathcal{P}(\mathbf{Sign}))$$

using the extraction function sign.

The independent attribute method gives

$$(\mathcal{P}(\mathbf{Z}) \times \mathcal{P}(\mathbf{Z}), \alpha_{\mathsf{SS}}, \gamma_{\mathsf{SS}}, \mathcal{P}(\mathbf{Sign}) \times \mathcal{P}(\mathbf{Sign}))$$

where

$$\begin{array}{rcl} \alpha_{\mathsf{SS}}(Z_1, Z_2) & = & (\{\mathsf{sign}(z) \mid z \in Z_1\}, \{\mathsf{sign}(z) \mid z \in Z_2\}) \\ \gamma_{\mathsf{SS}}(S_1, S_2) & = & (\{z \mid \mathsf{sign}(z) \in S_1\}, \{z \mid \mathsf{sign}(z) \in S_2\}) \end{array}$$

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Motivating the Relational Method

The independent attribute method often leads to imprecision!

Semantics: The expression `(x,-x)` may have a value in

$$\{(z, -z) \mid z \in \mathbf{Z}\}$$

Analysis: When we use $\mathcal{P}(\mathbf{Z}) \times \mathcal{P}(\mathbf{Z})$ to represent sets of pairs of integers we cannot do better than representing $\{(z, -z) \mid z \in \mathbf{Z}\}$ by

$$(\mathbf{Z}, \mathbf{Z})$$

Hence the best property describing it will be

$$\alpha_{\mathsf{SS}}(\mathbf{Z}, \mathbf{Z}) = (\{\text{-}, 0, \text{+}\}, \{\text{-}, 0, \text{+}\})$$

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Relational Method

Let $(\mathcal{P}(V_1), \alpha_1, \gamma_1, \mathcal{P}(D_1))$ and $(\mathcal{P}(V_2), \alpha_2, \gamma_2, \mathcal{P}(D_2))$ be Galois connections.

The *relational method* will give rise to the Galois connection

$$(\mathcal{P}(V_1 \times V_2), \alpha, \gamma, \mathcal{P}(D_1 \times D_2))$$

where

$$\begin{aligned} \alpha(VV) &= \bigcup\{\alpha_1(\{v_1\}) \times \alpha_2(\{v_2\}) \mid (v_1, v_2) \in VV\} \\ \gamma(DD) &= \{(v_1, v_2) \mid \alpha_1(\{v_1\}) \times \alpha_2(\{v_2\}) \subseteq DD\} \end{aligned}$$

Generalisation to arbitrary complete lattices: use *tensor products*.

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Relational Method from Extraction Functions

Assume that the Galois connections $(\mathcal{P}(V_i), \alpha_i, \gamma_i, \mathcal{P}(D_i))$ are given by *extraction functions* $\eta_i : V_i \to D_i$ as in

$$\begin{aligned}\alpha_i(V_i') &= \{\eta_i(v_i) \mid v_i \in V_i'\}\\ \gamma_i(D_i') &= \{v_i \mid \eta_i(v_i) \in D_i'\}\end{aligned}$$

Then the Galois connection $(\mathcal{P}(V_1 \times V_2), \alpha, \gamma, \mathcal{P}(D_1 \times D_2))$ has

$$\begin{aligned}\alpha(VV) &= \{(\eta_1(v_1), \eta_2(v_2)) \mid (v_1, v_2) \in VV\}\\ \gamma(DD) &= \{(v_1, v_2) \mid (\eta_1(v_1), \eta_2(v_2)) \in DD\}\end{aligned}$$

which also can be obtained directly from the extraction function $\eta : V_1 \times V_2 \to D_1 \times D_2$ defined by

$$\eta(v_1, v_2) = (\eta_1(v_1), \eta_2(v_2))$$

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Example: Detection of Signs Analysis

Using the relational method we get a Galois connection

$$(\mathcal{P}(\mathbf{Z} \times \mathbf{Z}), \alpha_{\mathsf{SS}'}, \gamma_{\mathsf{SS}'}, \mathcal{P}(\mathbf{Sign} \times \mathbf{Sign}))$$

where

$$\begin{array}{rcl} \alpha_{\mathsf{SS}'}(ZZ) & = & \{(\mathsf{sign}(z_1), \mathsf{sign}(z_2)) \mid (z_1, z_2) \in ZZ\} \\ \gamma_{\mathsf{SS}'}(SS) & = & \{(z_1, z_2) \mid (\mathsf{sign}(z_1), \mathsf{sign}(z_2)) \in SS\} \end{array}$$

corresponding to an extraction function $\mathsf{twosigns} : \mathbf{Z} \times \mathbf{Z} \rightarrow \mathbf{Sign} \times \mathbf{Sign}$ defined by

$$\mathsf{twosigns}(z_1, z_2) = (\mathsf{sign}(z_1), \mathsf{sign}(z_2))$$

Advantages of the Relational Method

Semantics: The expression `(x,-x)` may have a value in

$$\{(z, -z) \mid z \in \mathbf{Z}\}$$

In the present setting $\{(z, -z) \mid z \in \mathbf{Z}\}$ is an element of $\mathcal{P}(\mathbf{Z} \times \mathbf{Z})$.

Analysis: The best "relational" property describing it is

$$\alpha_{\mathsf{SS}'}(\{(z, -z) \mid z \in \mathbf{Z}\}) = \{(\mathsf{-}, \mathsf{+}), (\mathsf{0}, \mathsf{0}), (\mathsf{+}, \mathsf{-})\}$$

whereas the best "independent attribute" property was

$$\alpha_{\mathsf{SS}}(\mathbf{Z}, \mathbf{Z}) = (\{\mathsf{-}, \mathsf{0}, \mathsf{+}\}, \{\mathsf{-}, \mathsf{0}, \mathsf{+}\})$$

Function Spaces

Total Function Space

Let (L, α, γ, M) be a Galois connection and let S be a set.

The Galois connection for the *total function space*

$$(S \to L, \alpha', \gamma', S \to M)$$

is defined by

$$\alpha'(f) = \alpha \circ f \qquad\qquad \gamma'(g) = \gamma \circ g$$

Do we need to assume that S is non-empty?

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Monotone Function Space

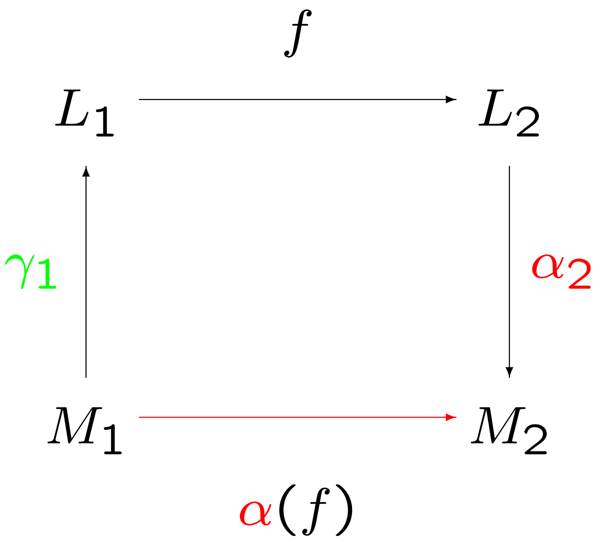
Let $(L_1, \alpha_1, \gamma_1, M_1)$ and $(L_2, \alpha_2, \gamma_2, M_2)$ be Galois connections.

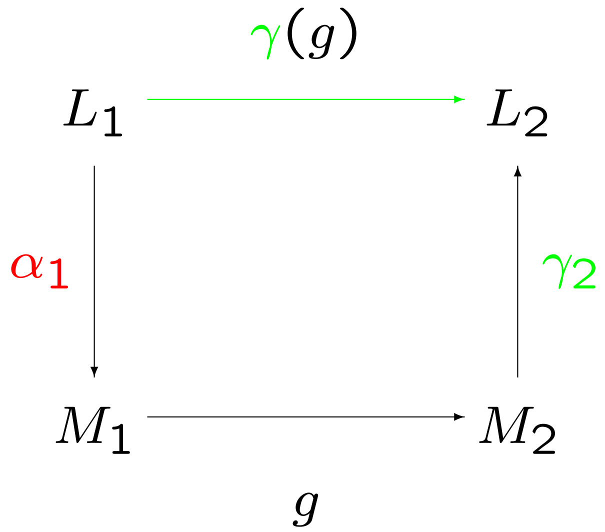
The Galois connection for the *monotone function space*

$$(L_1 \to L_2, \alpha, \gamma, M_1 \to M_2)$$

is defined by

$$\alpha(f) = \alpha_2 \circ f \circ \gamma_1 \qquad \gamma(g) = \gamma_2 \circ g \circ \alpha_1$$





Performing Analyses Simultaneously

Direct Product

Let $(L, \alpha_1, \gamma_1, M_1)$ and $(L, \alpha_2, \gamma_2, M_2)$ be Galois connections.

The *direct product* is the Galois connection

$$(L, \alpha, \gamma, M_1 \times M_2)$$

defined by

$$\begin{aligned} \alpha(l) &= (\alpha_1(l), \alpha_2(l)) \\ \gamma(m_1, m_2) &= \gamma_1(m_1) \sqcap \gamma_2(m_2) \end{aligned}$$

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Example:

Combining the detection of signs analysis for pairs of integers with the analysis of difference in magnitude.

We get the Galois connection

$$(\mathcal{P}(\mathbf{Z} \times \mathbf{Z}), \alpha_{\mathsf{SSR}}, \gamma_{\mathsf{SSR}}, \mathcal{P}(\mathbf{Sign} \times \mathbf{Sign}) \times \mathcal{P}(\mathbf{Range}))$$

where

$$\begin{array}{rcl} \alpha_{\mathsf{SSR}}(ZZ) & = & (\{(\mathsf{sign}(z_1), \mathsf{sign}(z_2)) \mid (z_1, z_2) \in ZZ\}, \\ & & \{\mathsf{range}(|z_1| - |z_2|) \mid (z_1, z_2) \in ZZ\}) \\ \gamma_{\mathsf{SSR}}(SS, R) & = & \{(z_1, z_2) \mid (\mathsf{sign}(z_1), \mathsf{sign}(z_2)) \in SS\} \\ & \cap & \{(z_1, z_2) \mid \mathsf{range}(|z_1| - |z_2|) \in R\} \end{array}$$

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Motivating the Direct Tensor Product

The expression `(x, 3*x)` may have a value in

$$\{(z, 3*z) \mid z \in \mathbf{Z}\}$$

which is described by

$$\alpha_{\mathsf{SSR}}(\{(z, 3*z) \mid z \in \mathbf{Z}\}) = (\{(\mathsf{-},\mathsf{-}),(\mathsf{0},\mathsf{0}),(\mathsf{+},\mathsf{+})\},\{\mathsf{0},\mathsf{<\text{-}1}\})$$

But

- any pair described by $(\mathsf{0},\mathsf{0})$ will have a difference in magnitude described by 0
- any pair described by $(\mathsf{-},\mathsf{-})$ or $(\mathsf{+},\mathsf{+})$ will have a difference in magnitude described by <-1

and the analysis cannot express this.

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Direct Tensor Product

Let $(\mathcal{P}(V), \alpha_1, \gamma_1, \mathcal{P}(D_1))$ and $(\mathcal{P}(V), \alpha_2, \gamma_2, \mathcal{P}(D_2))$ be Galois connections.

The *direct tensor product* is the Galois connection

$$(\mathcal{P}(V), \alpha, \gamma, \mathcal{P}(D_1 \times D_2))$$

defined by

$$\begin{aligned} \alpha(V') &= \bigcup\{\alpha_1(\{v\}) \times \alpha_2(\{v\}) \mid v \in V'\} \\ \gamma(DD) &= \{v \mid \alpha_1(\{v\}) \times \alpha_2(\{v\}) \subseteq DD\} \end{aligned}$$

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Direct Tensor Product from Extraction Functions

Assume that the Galois connections $(\mathcal{P}(V), \alpha_i, \gamma_i, \mathcal{P}(D_i))$ are given by *extraction functions* $\eta_i : V \to D_i$ as in

$$\begin{aligned} \alpha_i(V') &= \{\eta_i(v) \mid v \in V'\} \\ \gamma_i(D_i') &= \{v \mid \eta_i(v) \in D_i'\} \end{aligned}$$

The Galois connection $(\mathcal{P}(V), \alpha, \gamma, \mathcal{P}(D_1 \times D_2))$ has

$$\begin{aligned} \alpha(V') &= \{(\eta_1(v), \eta_2(v)) \mid v \in V'\} \\ \gamma(DD) &= \{v \mid (\eta_1(v), \eta_2(v)) \in DD\} \end{aligned}$$

corresponding to the extraction function $\eta : V \to D_1 \times D_2$ defined by

$$\eta(v) = (\eta_1(v), \eta_2(v))$$

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Example:

Using the direct tensor product to combine the detection of signs analysis for pairs of integers with the analysis of difference in magnitude.

$$(\mathcal{P}(\mathbf{Z}\times\mathbf{Z}), \alpha_{\mathsf{SSR}'}, \gamma_{\mathsf{SSR}'}, \mathcal{P}(\mathbf{Sign}\times\mathbf{Sign}\times\mathbf{Range}))$$

is given by

$$\begin{aligned}\alpha_{\mathsf{SSR}'}(ZZ) &= \{(\mathsf{sign}(z_1), \mathsf{sign}(z_2), \mathsf{range}(|z_1|-|z_2|)) \mid (z_1, z_2)\in ZZ\}\\ \gamma_{\mathsf{SSR}'}(SSR) &= \{(z_1, z_2) \mid (\mathsf{sign}(z_1), \mathsf{sign}(z_2), \mathsf{range}(|z_1|-|z_2|)) \in SSR\}\end{aligned}$$

corresponding to twosignsrange $: \mathbf{Z}\times\mathbf{Z}\to\mathbf{Sign}\times\mathbf{Sign}\times\mathbf{Range}$ given by

$$\mathsf{twosignsrange}(z_1, z_2) = (\mathsf{sign}(z_1), \mathsf{sign}(z_2), \mathsf{range}(|z_1|-|z_2|))$$

Advantages of the Direct Tensor Product

The expression `(x,3*x)` may have a value in $\{(z, 3*z) \mid z \in \mathbf{Z}\}$ which in the direct tensor product can be described by

$$\alpha_{\mathsf{SSR'}}(\{(z, 3*z) \mid z \in \mathbf{Z}\}) = \{(\mathsf{-}, \mathsf{-}, \mathsf{<\text{-}1}), (\mathsf{0}, \mathsf{0}, \mathsf{0}), (\mathsf{+}, \mathsf{+}, \mathsf{<\text{-}1})\}$$

compared to the direct product that gave

$$\alpha_{\mathsf{SSR}}(\{(z, 3*z) \mid z \in \mathbf{Z}\}) = (\{(\mathsf{-}, \mathsf{-}), (\mathsf{0}, \mathsf{0}), (\mathsf{+}, \mathsf{+})\}, \{\mathsf{0}, \mathsf{<\text{-}1}\})$$

Note that the Galois connection is *not* a Galois insertion because

$$\gamma_{\mathsf{SSR'}}(\emptyset) = \emptyset = \gamma_{\mathsf{SSR'}}(\{(\mathsf{0}, \mathsf{0}, \mathsf{<\text{-}1})\})$$

so $\gamma_{\mathsf{SSR'}}$ is not injective and hence we do not have a Galois insertion.

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From Direct to Reduced

Reduced Product

Let $(L, \alpha_1, \gamma_1, M_1)$ and $(L, \alpha_2, \gamma_2, M_2)$ be Galois connections.

The *reduced product* is the Galois *insertion*

$$(L, \alpha, \gamma, \varsigma[M_1 \times M_2])$$

defined by

$$\begin{aligned} \alpha(l) &= (\alpha_1(l), \alpha_2(l)) \\ \gamma(m_1, m_2) &= \gamma_1(m_1) \sqcap \gamma_2(m_2) \\ \varsigma(m_1, m_2) &= \bigsqcap\{(m_1', m_2') \mid \gamma_1(m_1) \sqcap \gamma_2(m_2) = \gamma_1(m_1') \sqcap \gamma_2(m_2')\} \end{aligned}$$

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Reduced Tensor Product

Let $(\mathcal{P}(V), \alpha_1, \gamma_1, \mathcal{P}(D_1))$ and $(\mathcal{P}(V), \alpha_2, \gamma_2, \mathcal{P}(D_2))$ be Galois connection.

The *reduced tensor product* is the Galois *insertion*

$$(\mathcal{P}(V), \alpha, \gamma, \varsigma[\mathcal{P}(D_1 \times D_2)])$$

defined by

$$\begin{array}{rcl}
\alpha(V') & = & \bigcup\{\alpha_1(\{v\}) \times \alpha_2(\{v\}) \mid v \in V'\} \\
\gamma(\mathit{DD}) & = & \{v \mid \alpha_1(\{v\}) \times \alpha_2(\{v\}) \subseteq \mathit{DD}\} \\
\varsigma(\mathit{DD}) & = & \bigcap\{\mathit{DD}' \mid \gamma(\mathit{DD}) = \gamma(\mathit{DD}')\}
\end{array}$$

Example: Array Bounds Analysis

The superfluous elements of $\mathcal{P}(\mathbf{Sign} \times \mathbf{Sign} \times \mathbf{Range})$ will be removed when we use a reduced tensor product:

The reduction operator $\varsigma_{\mathsf{SSR}'}$ amounts to

$$\varsigma_{\mathsf{SSR}'}(SSR) = \bigcap\{SSR' \mid \gamma_{\mathsf{SSR}'}(SSR) = \gamma_{\mathsf{SSR}'}(SSR')\}$$

where $SSR, SSR' \subseteq \mathbf{Sign} \times \mathbf{Sign} \times \mathbf{Range}$.

The singleton sets constructed from the following 16 elements

$$\begin{array}{llll}
(-, 0, \texttt{<-1}), & (-, 0, -1), & (-, 0, 0), & \\
(0, -, 0), & (0, -, +1), & (0, -, \texttt{>+1}), & \\
(0, 0, \texttt{<-1}), & (0, 0, -1), & (0, 0, +1), & (0, 0, \texttt{>+1}), \\
(0, +, 0), & (0, +, +1), & (0, +, \texttt{>+1}), & \\
(+, 0, \texttt{<-1}), & (+, 0, -1), & (+, 0, 0) &
\end{array}$$

will be mapped to the empty set (as they are useless).

Example (cont.): Array Bounds Analysis

The remaining 29 elements of $\mathbf{Sign} \times \mathbf{Sign} \times \textbf{Range}$ are

$$
\begin{array}{lllll}
(-,-,<\!-1), & (-,-,-1), & (-,-,0), & (-,-,+1), & (-,-,>\!+1),\\
(-,0,+1), & (-,0,>\!+1), & & & \\
(-,+,<\!-1), & (-,+,-1), & (-,+,0), & (-,+,+1), & (-,+,>\!+1),\\
(0,-,<\!-1), & (0,-,-1), & (0,0,0), & (0,+,<\!-1), & (0,+,-1),\\
(+,-,<\!-1), & (+,-,-1), & (+,-,0), & (+,-,+1), & (+,-,>\!+1),\\
(+,0,+1), & (+,0,>\!+1), & & & \\
(+,+,<\!-1), & (+,+,-1), & (+,+,0), & (+,+,+1), & (+,+,>\!+1)
\end{array}
$$

and they describe disjoint subsets of $\mathbf{Z} \times \mathbf{Z}$.

Any collection of properties can be descibed in 4 bytes.

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Summary

The Array Bound Analysis has been designed from three simple Galois connections specified by extraction functions:

(i) an analysis approximating integers by their sign,

(ii) an analysis approximating pairs of integers by their difference in magnitude, and

(iii) an analysis approximating integers by their closeness to 0, 1 and -1.

These analyses have been combined using:

(iv) the relational product of analysis (i) with itself,

(v) the functional composition of analyses (ii) and (iii), and

(vi) the reduced tensor product of analyses (iv) and (v).

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Induced Operations

Given: Galois connections $(L_i, \alpha_i, \gamma_i, M_i)$ so that M_i is more approximate than (i.e. is coarser than) L_i.

Aim: Replace an existing analysis over L_i with an analysis making use of the coarser structure of M_i.

Methods:

- Inducing along the abstraction function: move the computations from L_i to M_i.
- Application to Data Flow Analysis.
- Inducing along the concretisation function: move a widening from M_i to L_i.

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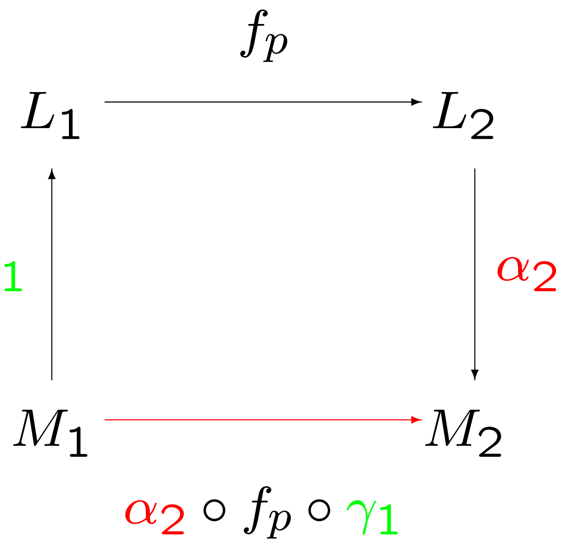
Inducing along the Abstraction Function

Given Galois connections $(L_i, \alpha_i, \gamma_i, M_i)$ so that M_i is more approximate than L_i.

Replace an existing analysis $f_p : L_1 \to L_2$ with a new and more approximate analysis $g_p : M_1 \to M_2$: take $g_p = \alpha_2 \circ f_p \circ \gamma_1$.



The analysis $\alpha_2 \circ f_p \circ \gamma_1$ is *induced* from f_p and the Galois connections.

Example:

A very precise analysis for **plus** based on $\mathcal{P}(\mathbf{Z})$ and $\mathcal{P}(\mathbf{Z} \times \mathbf{Z})$:

$$f_{\mathsf{plus}}(ZZ) = \{z_1 + z_2 \mid (z_1, z_2) \in ZZ\}$$

Two Galois connections

$$(\mathcal{P}(\mathbf{Z}), \alpha_{\mathsf{sign}}, \gamma_{\mathsf{sign}}, \mathcal{P}(\mathbf{Sign}))$$

$$(\mathcal{P}(\mathbf{Z} \times \mathbf{Z}), \alpha_{\mathsf{SS'}}, \gamma_{\mathsf{SS'}}, \mathcal{P}(\mathbf{Sign} \times \mathbf{Sign}))$$

An approximate analysis for **plus** based on $\mathcal{P}(\mathbf{Sign})$ and $\mathcal{P}(\mathbf{Sign} \times \mathbf{Sign})$:

$$g_{\mathsf{plus}} = \alpha_{\mathsf{sign}} \circ f_{\mathsf{plus}} \circ \gamma_{\mathsf{SS'}}$$

Example (cont.):

We calculate

$$
\begin{aligned}
g_{\mathsf{plus}}(SS) &= \alpha_{\mathsf{sign}}(f_{\mathsf{plus}}(\gamma_{\mathsf{SS'}}(SS))) \\
&= \alpha_{\mathsf{sign}}(f_{\mathsf{plus}}(\{(z_1, z_2) \in \mathbf{Z} \times \mathbf{Z} \mid (\mathsf{sign}(z_1), \mathsf{sign}(z_2)) \in SS\})) \\
&= \alpha_{\mathsf{sign}}(\{z_1 + z_2 \mid z_1, z_2 \in \mathbf{Z}, (\mathsf{sign}(z_1), \mathsf{sign}(z_2)) \in SS\}) \\
&= \{\mathsf{sign}(z_1 + z_2) \mid z_1, z_2 \in \mathbf{Z}, (\mathsf{sign}(z_1), \mathsf{sign}(z_2)) \in SS\} \\
&= \bigcup\{s_1 \oplus s_2 \mid (s_1, s_2) \in SS\}
\end{aligned}
$$

where $\oplus : \mathbf{Sign} \times \mathbf{Sign} \rightarrow \mathcal{P}(\mathbf{Sign})$ is the "addition" operator on signs (so e.g. $\mathsf{+} \oplus \mathsf{+} = \{\mathsf{+}\}$ and $\mathsf{+} \oplus \mathsf{-} = \{\mathsf{-}, \mathsf{0}, \mathsf{+}\}$).

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The Mundane Correctness of f_p carries over to g_p

The correctness relation R_i for V_i and L_i:

$$R_i : V_i \times L_i \to \{\mathit{true}, \mathit{false}\} \text{ is } \textit{generated by } \beta_i : V_i \to L_i$$

Correctness of f_p means

$$(p \vdash \cdot \leadsto \cdot)\,(R_1 \twoheadrightarrow R_2)\, f_p$$

(with $R_1 \twoheadrightarrow R_2$ being generated by $\beta_1 \twoheadrightarrow \beta_2$).

The correctness relation S_i for V_i and M_i:

$$S_i : V_i \times M_i \to \{\mathit{true}, \mathit{false}\} \text{ is } \textit{generated by } \alpha_i \circ \beta_i : V_i \to M_i$$

One can prove that

$$(p \vdash \cdot \leadsto \cdot)\,(R_1 \twoheadrightarrow R_2)\, f_p \;\wedge\; \alpha_2 \circ f_p \circ \gamma_1 \sqsubseteq g_p$$
$$\Rightarrow\; (p \vdash \cdot \leadsto \cdot)\,(S_1 \twoheadrightarrow S_2)\, g_p$$

with $S_1 \twoheadrightarrow S_2$ being generated by $(\alpha_1 \circ \beta_1) \twoheadrightarrow (\alpha_2 \circ \beta_2)$.

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Fixed Points in the Induced Analysis

Let $f_p = \mathit{lfp}(F)$ for a monotone function $F : (L_1 \to L_2) \to (L_1 \to L_2)$.

The Galois connections $(L_i, \alpha_i, \gamma_i, M_i)$ give rise to a Galois connection $(L_1 \to L_2, \alpha, \gamma, M_1 \to M_2)$.

Take $g_p = \mathit{lfp}(G)$ where $G : (M_1 \to M_2) \to (M_1 \to M_2)$ is an "upper approximation" to F: we demand that $\alpha \circ F \circ \gamma \sqsubseteq G$.

Then for all $m \in M_1 \to M_2$:

$$G(m) \sqsubseteq m \Rightarrow F(\gamma(m)) \sqsubseteq \gamma(m)$$

and $\mathit{lfp}(F) \sqsubseteq \gamma(\mathit{lfp}(G))$ and $\alpha(\mathit{lfp}(F)) \sqsubseteq \mathit{lfp}(G)$

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Application to Data Flow Analysis

A *generalised Monotone Framework* consists of:

- the property space: a complete lattice $L = (L, \sqsubseteq)$;
- the set $\mathcal{F}$ of monotone functions from L to L.

An *instance* A of a generalised Monotone Framework consists of:

- a finite flow, $F \subseteq \mathbf{Lab} \times \mathbf{Lab}$;
- a finite set of extremal labels, $E \subseteq \mathbf{Lab}$;
- an extremal value, $\iota \in L$; and
- a mapping $f_\cdot$ from the labels $\mathbf{Lab}$ of F and E to monotone transfer functions from L to L.

Application to Data Flow Analysis

Let (L, α, γ, M) be a Galois connection.

Consider an instance B of the generalised Monotone Framework M that satisfies

- the mapping $g_\cdot$ from the labels $\mathbf{Lab}$ of F and E to monotone transfer functions of $M \to M$ satisfies $g_\ell \sqsupseteq \alpha \circ f_\ell \circ \gamma$ for all ℓ; and
- the extremal value $\jmath$ satisfies $\gamma(\jmath) = \iota$;

and otherwise B is as A.

One can show that a solution to the B-constraints gives rise to a solution to the A-constraints:

$$(B_\circ, B_\bullet) \models \mathsf{B}^{\sqsupseteq} \quad \text{implies} \quad (\gamma \circ B_\circ, \gamma \circ B_\bullet) \models \mathsf{A}^{\sqsupseteq}$$

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The Mundane Approach to Semantic Correctness

Here $F = \mathit{flow}(S_\star)$ and $E = \{\mathit{init}(S_\star)\}$.

Correctness of every solution to $\mathsf{A}^{\sqsupseteq}$ amounts to:

Assume $(A_\circ, A_\bullet) \models \mathsf{A}^{\sqsupseteq}$ and $\langle S_\star, \sigma_1 \rangle \to^* \sigma_2$.

Then $\beta(\sigma_1) \sqsubseteq \iota$ implies $\beta(\sigma_2) \sqsubseteq \bigsqcup\{A_\bullet(\ell) \mid \ell \in \mathit{final}(S_\star)\}$.

where $\beta : \mathbf{State} \to L$.

One can then prove the correctness result for B:

Assume $(B_\circ, B_\bullet) \models \mathsf{B}^{\sqsupseteq}$ and $\langle S_\star, \sigma_1 \rangle \to^* \sigma_2$.

Then $(\alpha \circ \beta)(\sigma_1) \sqsubseteq \jmath$ implies $(\alpha \circ \beta)(\sigma_2) \sqsubseteq \bigsqcup\{B_\bullet(\ell) \mid \ell \in \mathit{final}(S_\star)\}$.

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Sets of States Analysis

Generalised Monotone Framework over $(\mathcal{P}(\mathbf{State}), \subseteq)$.
Instance SS for $S_\star$:

- the flow F is *flow*$(S_\star)$;
- the set E of extremal labels is $\{$*init*$(S_\star)\}$;
- the extremal value ι is $\mathbf{State}$; and
- the transfer functions are given by $f_\cdot^{\mathsf{SS}}$:

$$\begin{array}{lrcl} [x := a]^\ell & f_\ell^{\mathsf{SS}}(\Sigma) & = & \{\sigma[x \mapsto \mathcal{A}[\![a]\!]\sigma] \mid \sigma \in \Sigma\} \\ [\texttt{skip}]^\ell & f_\ell^{\mathsf{SS}}(\Sigma) & = & \Sigma \\ [b]^\ell & f_\ell^{\mathsf{SS}}(\Sigma) & = & \Sigma \end{array}$$

where $\Sigma \subseteq \mathbf{State}$.

Correctness: Assume $(\mathit{SS}_\circ, \mathit{SS}_\bullet) \models \mathsf{SS}^{\supseteq}$ and $\langle S_\star, \sigma_1 \rangle \to^* \sigma_2$.
Then $\sigma_1 \in \mathbf{State}$ implies $\sigma_2 \in \bigcup\{\mathit{SS}_\bullet(\ell) \mid \ell \in$ *final*$(S_\star)\}$.

Constant Propagation Analysis

Generalised Monotone Framework over $\widehat{\mathbf{State}}_{\mathsf{CP}} = ((\mathbf{Var} \to \mathbf{Z}^{\top})_{\perp}, \sqsubseteq)$.
Instance CP for $S_\star$:

- the flow F is *flow*$(S_\star)$;
- the set E of extremal labels is $\{$*init*$(S_\star)\}$;
- the extremal value ι is $\lambda x.\top$; and
- the transfer functions are given by the mapping $f_{\cdot}^{\mathsf{CP}}$:

$$
\begin{array}{lrcl}
[x := a]^\ell: & f_\ell^{\mathsf{CP}}(\widehat{\sigma}) & = & \begin{cases} \perp & \text{if } \widehat{\sigma} = \perp \\ \widehat{\sigma}[x \mapsto \mathcal{A}_{\mathsf{CP}}[\![a]\!]\widehat{\sigma}] & \text{otherwise} \end{cases} \\
[\texttt{skip}]^\ell: & f_\ell^{\mathsf{CP}}(\widehat{\sigma}) & = & \widehat{\sigma} \\
[b]^\ell: & f_\ell^{\mathsf{CP}}(\widehat{\sigma}) & = & \widehat{\sigma}
\end{array}
$$

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Galois Connection

The representation function $\beta_{\mathsf{CP}} : \mathbf{State} \to \widehat{\mathbf{State}}_{\mathsf{CP}}$ is defined by

$$\beta_{\mathsf{CP}}(\sigma) = \sigma$$

This gives rise to a Galois connection

$$(\mathcal{P}(\mathbf{State}), \alpha_{\mathsf{CP}}, \gamma_{\mathsf{CP}}, \widehat{\mathbf{State}}_{\mathsf{CP}})$$

where $\alpha_{\mathsf{CP}}(\Sigma) = \bigsqcup\{\beta_{\mathsf{CP}}(\sigma) \mid \sigma \in \Sigma\}$ and $\gamma_{\mathsf{CP}}(\widehat{\sigma}) = \{\sigma \mid \beta_{\mathsf{CP}}(\sigma) \sqsubseteq \widehat{\sigma}\}$.

One can show that for all labels ℓ

$f_\ell^{\mathsf{CP}} \sqsupseteq \alpha_{\mathsf{CP}} \circ f_\ell^{\mathsf{SS}} \circ \gamma_{\mathsf{CP}}$ as well as $\gamma_{\mathsf{CP}}(\lambda x.\top) = \mathbf{State}$

It follows that CP is an upper approximation to the analysis induced from SS and the Galois connection; therefore it is correct.

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Inducing along the Concretisation Function

Given an upper bound operator

$$\nabla_M : M \times M \to M$$

and a Galois connection (L, α, γ, M).

Define an upper bound operator

$$\nabla_L : L \times L \to L$$

by

$$l_1 \ \nabla_L \ l_2 = \gamma(\ \alpha(l_1) \ \nabla_M \ \alpha(l_2) \ )$$

It defines a widening operator if one of the following conditions holds:

(i) M satisfies the Ascending Chain Condition, or

(ii) (L, α, γ, M) is a Galois insertion and $\nabla_M : M \times M \to M$ is a widening.

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Precision of the Induced Widening Operator

Lemma: Let (L, α, γ, M) be a Galois insertion such that $\gamma(\bot_M) = \bot_L$ and let $\nabla_M : M \times M \to M$ be a widening operator.

Then the widening operator $\nabla_L : L \times L \to L$ defined by

$$l_1 \ \nabla_L \ l_2 = \gamma(\alpha(l_1) \ \nabla_M \ \alpha(l_2))$$

satisfies

$$\mathit{lfp}_{\nabla_L}(f) = \gamma(\mathit{lfp}_{\nabla_M}(\alpha \circ f \circ \gamma))$$

for all monotone functions $f : L \to L$.

Precision of the Induced Widening Operator

Corollary: Let M be of finite height, let (L, α, γ, M) be a Galois insertion (such that $\gamma(\bot_M) = \bot_L$), and let ∇_M equal the least upper bound operator $\sqcup_M$.

Then the above lemma shows that $\mathit{lfp}_{\nabla_L}(f) = \gamma(\mathit{lfp}(\alpha \circ f \circ \gamma))$.

This means that $\mathit{lfp}_{\nabla_L}(f)$ *equals* the result we would have obtained if we decided to work with $\alpha \circ f \circ \gamma : M \to M$ instead of the given $f : L \to L$; furthermore the number of iterations needed turn out to be the same. However, for all other operations the increased precision of L is available.

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